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# The Sixth and Seventh Largest Number of Subuniverses of Finite Lattices

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## ABSTRACT

By a subuniverse, we mean a sublattice or the empty- set. We prove that the sixth largest number of subuniverses of an $n$-element lattice is $21.125 \cdot 2^{n-5}$ and the seventh largest number is $20.75 \cdot 2^{n-5}$. Also, we describe the $n$-element lattices with exactly $21.125 \cdot 2^{n-5}$ and $20.75 \cdot 2^{n-5}$ subuniverses.

KEYWORDS: Finite lattices, subuniverses, sublattice, number of sublattices, partial sublattice

## 1. INTRODUCTION

Let $L$ be finite lattice, Sub($L$) will denote its *sublattice lattice*; Sub($L$) consists of all *subuniverses* of $L$. A subset $X$ of $L$ is in Sub($L$) if and only if $X$ is closed with respect to join and meet. Note that $\emptyset \in$ Sub($L$); moreover for $X \in$ Sub($L$), $X$ is a sublattice of $L$ iff $X$ is nonempty. This work is a natural continuation of (Ahmed et al, 2019) and (Cz´edli and Horv´ath, 2019), where the first fifth largest numbers of subuniverses have been determined. To read more on similar work see the bibliography indicated in (Cz´edli, 2018; Cz´edli, 2019a; Cz´edli, 2019b; Cz´edli, 2019c; Cz´edli and Horv´ath, 2019; Kulin and Muresan (2018) and Freese (1997).

For basic lattice theory see e.g. Gr¨atzer (2011), We recall some notions and tools from [4] and [6]. An element $u \in L$ *isolated* if $u \in L \setminus \{0, 1\}$ has a unique lower cover and a unique upper cover, and, in addition, $x \parallel u$ holds for no $x \in L$. An interval $[u, v]$ will be called an *isolated edge* if $u \prec v$, and $L = {\downarrow}u \cup {\uparrow}v$. The next lemma is from [6], and we will use it very often in this paper.

Lemma 1.1. (Cz´edli and Horv´ath, 2019) If K is a sublattice and H is a subset of a finite lattice
L, then the following three assertions hold.

i) With the notation $t := |\{H \cap S : S \in \mathrm{Sub}(L)\}|$, we have that $|\mathrm{Sub}(L)| \le t \cdot 2^{|L|-|H|}$.

ii) $|\mathrm{Sub}(L)| \le |\mathrm{Sub}(K)| \cdot 2^{|L|-|K|}$.

iii) Assume, in addition, that K has neither an isolated element, nor an isolated edge. Then $|\mathrm{Sub}(L)| = |\mathrm{Sub}(K)| \cdot 2^{|L|-|K|}$ if and only if L is (isomorphic to) $C_0 +_{glu} K +_{glu} C_1$ for some chains $C_0$ and $C_1$.

Let $S = (S; \vee_S, \wedge_S)$ be a partial lattice; we use this term here to mean that S is a partial algebra with two binary operations. A sub- universe of S is a subset Y of S such that whenever $a, b \in Y$ and $a \vee_S b$ is defined in S, then $a \vee_S b \in Y$, and the same is true for $\wedge_S$.

We say that the partial lattice S is a partial sublattice of the lattice $L = (L; \vee_L, \wedge_L)$, if S is a subposet of L and whenever $a \parallel b$ for $a, b \in S$ and their join $a \vee_S b$ exists, then $a \vee_S b = a \vee_L b$, and the same is true for

$\wedge_S$. Without any danger of confusion, from now on we use the notation $L$ for a lattice (and $S$ for a partial lattice) again. In order to give an example for a partial lattice we define $H_1$ which will be used latter, it is the seven-element partial lattice $\{o, i, a, b, c, v, d\}$ with the condition

$\{o, i, a, b, c\} \parallel d$, $a \vee b = c \vee b = i$, $a \wedge b = c \wedge b = o$ and $d \vee i = v$. See Fig. 1.
We need to recall the follwoing Lemma 3.3 from [6]; and lemma 1.3 from [1]; we wrote it down here in row:

Lemma 1.2. (Cz´edli and Horv´ath, 2019) *If an n-element lattice L has a 3-antichain, then we have that* $|\mathrm{Sub}(S)| \le 20 \cdot 2^{n-5}$.

Lemma 1.3. [1] *If* $|L| = n$ *for the lattice* $L$ *and* $S$ *is a partial sublattice of* $L$ *with* $|S| = k$ *and with* $|\mathrm{Sub}(S)| = m$, *then* $|\mathrm{Sub}(L)| \leq m \cdot 2^{n-k}$.

## 2. A PREPARATORY LEMMA

Lemma 2.1. For the lattices and a partial lattice given in Figs. 1-5, the following assertions hold.

(i) | Sub(B4 +glu C(2) +glu B4)| = 169 = 21.125 $\cdot 2^{8-5}$,

(ii) | Sub(N5B4)| = 69 = 17.25 $\cdot 2^{7-5}$,

(iii) | Sub(C(2) × C(3))| = 38 = 19 $\cdot 2^{6-5}$,

(iv) | Sub(H1)| = 79 = 19.75 $\cdot 2^{8-5}$,

(v) | Sub(H2)| = 38 = 19 $\cdot 2^{6-5}$,

(vi) | Sub(H3)| = 142 = 17.75 $\cdot 2^{8-5}$,

(vii) | Sub(N7)| = 83 = 20.75 $\cdot 2^{7-5}$,

(viii) | Sub(N6B4)| = 132 = 16.5 $\cdot 2^{8-5}$,

(ix) | Sub(N6)| = 43 = 21.5 $\cdot 2^{6-5}$,

(x) | Sub(N ')| = 37 = 18.5 $\cdot 2^{6-5}$,

(xi) |Sub(B4)| = 13 = 26 $\cdot 2^{4-5}$,

(xii) |Sub(N ')| = 65 = 16.25 $\cdot 2^{7-5}$,

(xiii) |Sub(N7B4)| = 255 = 15.93 $\cdot 2^{9-5}$,

(xiv) |Sub(N 'B4)| = 1185 = 11.56 $\cdot 2^{9-5}$.

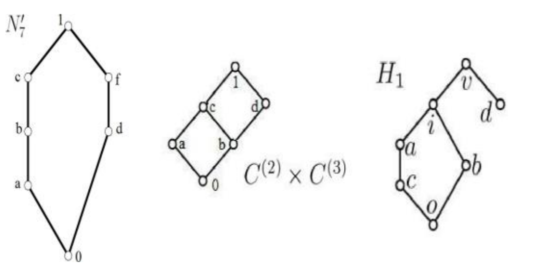


Fig. 1. N ', C(2) × C(3) and H1

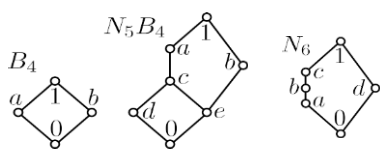


Fig. 2. B4, N5B4 and N6

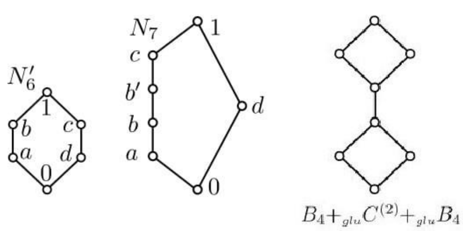


Fig. 3. N ', N7 and B4 +glu C(2) +glu B4

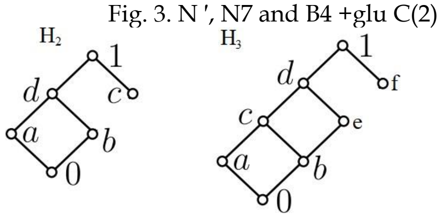


Fig. 4. H2 and H3

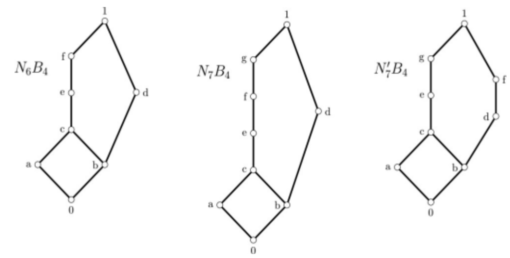


Fig. 5. N6B4

Proof. The notation given by Figs. 1-5, will be used. For the later reference, note that if L is a chain, then |Sub(L)| = 2|L|.

For (i), observe that

|{S ∈ Sub(B4 +glu C(2) +glu B4) : d ∉ S}| = 104, by (2.1)(iii)and (2.2)(i) of [6],

|{S ∈ Sub(B4 +glu C(2) +glu B4) : d ∈ S, {a, b, c, e, f } ∩ S = ∅}| = 4, and

|{S ∈ Sub(B4 +glu C(2) +glu B4) : d ∈ S, {a, b, c, e, f } ∩ S ≠ ∅}| = 61, whereby | Sub(B4 +glu C(2) +glu B4)| = 104 + 4 + 61 = 169 proves (i).

For (ii), observe that

|{S ∈ Sub(N5B4) : d ∉ S}| = 46, by 2.1 (iii) and 2.2 (ii) of [6],

|{S ∈ Sub(N5B4) : d ∈ S, b ∉ S}| = 20, and

|{S ∈ Sub(N5B4) : d ∈ S, b ∈ S}| = 3,

whereby | Sub(N5B4)| = 46 + 20 + 3 = 69 proves (ii).

For (iii), observe that

|{S ∈ Sub(C(2) × C(3)) : d ∉ S}| = 26, by 2.1 (iii) and 2.2 (ii) of [6],

|{S ∈ Sub(C(2) × C(3)) : d ∈ S, {a, b, c} ∩ S ≠ ∅}| = 8, and

|{S ∈ Sub(C(2) × C(3)) : {a, b} ∩ S = ∅}| = 4,

whereby | Sub(C(2) × C(3))| = 26 + 4 + 8 = 38 proves (iii).

For (iv), notice that H1 is a partial lattice, but not a lattice, so subuniverses are those subsets that are closed with respect to all partial operations, see also [4]. Observe that

|{S ∈ Sub(H1) : d ∉ S}| = 46, by 2.1 (iii) and 2.2 (ii) of [6],

|{S ∈ Sub(H1) : {d, v} ⊆ S}| = 23, and

the remaining subuniverses are the following: {b, d}, {o, b, d}, and all the elements of P ({o, a, c}) with d,

whereby | Sub(H1)| = 46 + 23 + 2 + 8 = 79 proves (iv).

For (viii), notice that

|{S ∈ Sub(H2) : c ∉ S}| = 26, by 2.1 (iii) and 2.2 (ii) of [6],

|{S ∈ Sub(H2) : {c} ⊆ S}| = 12,

whereby | Sub(H2)| = 12 + 26 = 38 proves (viii).

For (vi), notice that

|{S ∈ Sub(H3) : f ∉ S}| = 76, by 2.1 (iii) and 2.2 (ii) of

[6],

|{S ∈ Sub(H3) : {f } ⊆ S}| = 67,

whereby | Sub(H3)| = 76 + 67 = 142 proves (vi).

For (vii), observe that

|{S ∈ Sub(N7) : d ∉ S}| = 64, by (2.4) of [6],

|{S ∈ Sub(N7) : d ∈ S, {a, b, b′, c} ∩ S = ∅}| = 4, and

|{S ∈ Sub(N7) : d ∈ S, {a, b, b′, c} ∩ S ≠ ∅}| = 15,

whereby | Sub(N7)| = 64 + 4 + 15 = 83 proves (vii).

For (viii), observe that

|{S ∈ Sub(N6B4) : a ∉ S}| = 86, by lemma 2.1 (i) of [1], and 2.1(iii) of [6]

|{S ∈ Sub(N6B4) : a ∈ S, {b, c, e, d, f } ∩ S = ∅}| = 4, and

|{S ∈ Sub(N6B4) : a ∈ S, {b, c, e, d, f } ∩ S ≠ ∅}| = 42,

whereby | Sub(N6B4)| = 86 + 4 + 42 = 132 proves (viii).

For (ix), observe that

|{S ∈ Sub(N6) : d ∉ S}| = 32, by (2.4) of [6],

|{S ∈ Sub(N6) : d ∈ S, {a, b, c} ∩ S = ∅}| = 4, and

|{S ∈ Sub(N6) : d ∈ S, {a, b, c} ∩ S ≠ ∅}| = 7,

whereby | Sub(N6)| = 32 + 4 + 7 = 43 proves (ix).

For (x), observe that

|{S ∈ Sub(N ′) : c ∉ S}| = 23, by 2.1 (i) and 2.2 (ii) of [6],

|{S ∈ Sub(N ′) : d ∈ S, {a, b} ∩ S ≠ ∅}| = 6, and

|{S ∈ Sub(N ′) : {a, b} ∩ S = ∅}| = 8,

whereby | Sub(N ′)| = 23 + 6 + 8 = 37 proves (x).

For (xi), observe that

|{S ∈ Sub(B4) : b ∉ S}| = 8, (S is chain),

|{S ∈ Sub(B4) : b ∈ S, {a} ∩ S = ∅}| = 4, and

|{S ∈ Sub(B4) : b ∈ S, {a} ∩ S ≠ ∅}| = 1,

whereby | Sub(B4)| = 8 + 4 + 1 = 13 = 26 · 24−5 proves (xi).

For (xii), observe that

|{S ∈ Sub(N ′) : d ∉ S}| = 43, by 2.1 (ix),

|{S ∈ Sub(N ′) : d ∈ S, {a, b, c, f } ∩ S ≠ ∅}| = 18, and

|{S ∈ Sub(N ′) : d ∈ S, {a, b, c, f } ∩ S = ∅}| = 4, whereby | Sub(N7)| = 43 + 18 + 4 = 65 proves (xii).

For (xiii), observe that

|{S ∈ Sub(N7B4) : d ∉ S}| = 208, by 2.1 (iii) and 2.2 (ii) of [6],

|{S ∈ Sub(N7B4) : d ∈ S, {a, b, c, e, f, g} ∩ S ≠ ∅}| = 43, and

|{S ∈ Sub(N7B4) : d ∈ S, {a, b, c, e, f, g} ∩ S = ∅}| = 4,

whereby | Sub(N7B4)| = 208 + 43 + 4 = 255 proves (xiii).

For (xiv), observe that

|{S ∈ Sub(N ′B4) : f ∉ S}| = 132, by lemma 2.1 (viii)

|{S ∈ Sub(N ′B4) : f ∈ S, {a, b, c, e, d, g} ∩ S ≠ ∅}| = 49, and

|{S ∈ Sub(N ′B4) : f ∈ S, {a, b, c, e, d, g} ∩ S = ∅}| = 4, whereby | Sub(N ′B4)| = 132 + 49 + 4 = 185 proves (xiv).

Remark 2.2. A computer program is available for counting subuni- verses (and to prove the above Lemma) on the webpage of G. Cz´edli: http://www.math.u-szeged.hu/˜czedli/

## 3. THE MAIN RESULT

Following Cz´edli and Horv´ath (2019), for a natural number n ∈ N+ := {1, 2, 3, . . . }, let NS(n) := {| Sub(L)| : L is a lattice of size |L| = n}.

For further notions and notations see [4] and [6]. For the lattice N7,

see Figure 3. Our main result is the following

Theorem 3.1. If 6 ≤ n ∈ N+, then the following assertions hold.

(i) The sixth largest number in NS(n) is 21.125 ·2n−5. Furthermore, an n-element lattice L has exactly 21.125 · 2n−5 subuniverses if and only if L ∼= C0 +glu B4 +glu C2 +glu B4 +glu C1,where C0 , C1

and C2 are chains

(ii) The seventh largest number in NS(n) is 20.75 · 2n−5. Further- more, an n-element lattice L has exactly 20.75 · 2n−5 subuni- verses if and only if L ∼= C0 +glu N7 +glu C1, where C0 and C1

are chains.

A k-element antichain will be called a k-antichain, as in [6]. We also need the following well-known facts from the folklore.

Lemma 3.2. For every join-semilattice S generated by {a, b, c}, there is a unique surjective homomorphism φ from the free join-semilattice Fjsl(a˜, ˜b, c˜), given in Fig. 6, onto S such that φ(a˜) = a, φ(˜b) = b, and φ(c˜) = c.

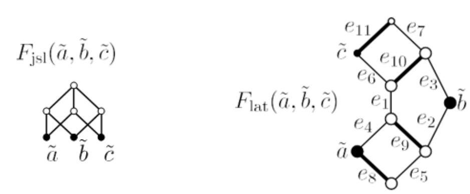


Fig. 6. $F_{jsl}$(a˜, ˜b, c˜) and $F_{lat}$(a˜, ˜b, c˜)

Lemma 3.3 (Rival and Wille [10, Figure 2]). For every lattice K generated by {a, b, c} such that a < c, there is a unique surjective ho- momorphism φ from the finitely presented lattice Flat(a˜, ˜b, c˜), given in Figure 6, onto K such that φ(a˜) = a, φ(˜b) = b, and φ(c˜) = c.

Proof of Theorem 3.1. We prove part (i);

Let L be an n-element lattice. We obtain from Lemma 1.1 (iii) and from 2.1(i) that if

L ∼= C0 +glu B4 +glu C2 +glu B4 +glu C1 for finite chains C0 and C1, (3.1) then | Sub(L) | = 21.125 · 2n−5. We know from [1] that the fifth largest number in NS(n) is 21.25 · 2n−5. Hence, in order to complete the proof of Theorem 3.1 (i), it suffices to exclude the existence of a lattice L such that

| L | = n, 21.125 · 2n−5 ≤ | Sub(L) | < 21.25 ·
2n−5, but L is not of the form given in (3.1). (3.2)

Suppose, for a contradiction, that L is a lattice satisfying (3.2). Then, by Theorem 1.1 of [6] and Lemma 3.3 of [6],

L has at least two 2-antichains but it has no 3-antichain. (3.3)

We state that

L cannot have two non-disjoint 2-antichains. (3.4)

Suppose to the contrary that {a, b} and {c, b} are two distinct 2- antichains in L. Since there is no 3-antichain in L, we can assume that a < c. With K := [{a, b, c}], let φ : Flat(a˜, ˜b, c˜) → K be the unique lattice homomorphism from Lemma 3.3, and let Θ be the kernel of φ. We follow the notations of Figure 6. If Θ does not collapse e1 and at least one of e4 or e6, then | Sub(L) | ≤ 17.25 · 2n−5 by Lemma 1.1 (ii)

and by Lemma 2.1(ii).

So, if Θ does not collapse e1, then it collapses both e4 and e6. Since in this case e1 also generates the monolith congruence of the N5 sub- lattice of Flat(a˜, ˜b, c˜), no other edge of this N5 sublattice is collapsed. Hence, N5 is a sublattice of L. Clearly, {a, b, c} generates a pentagon

N5. Keeping (3.2) in mind and applying Lemma 1.1 (iii) for K := N5, we obtain that L cannot be of form C0 +glu N5 +glu C1.

Let o and i stand for the least and the largest elements of the men- tioned N5 sublattice, respectively. By Lemma 2.1(iii), we can exclude that

↓o is a chain, ↑i is a chain, and [o, i] = N5. (3.5)

Thus, at least one of the three parts of (3.5) fails.

If ↓o is not a chain, then we would have a sublattice of form either

B4 +glu B4 or B4 +glu C1 +glu B4, but then the number of subuniverses could be at most 21.25 · 2n−5. By Lemma 1.1 (ii), moreover 21.25 · 2n−5 can appear only in case that L is of form C0 +glu B4 +glu B4 +glu C1. Hence, ↓o is a chain. We obtain, by duality, for later reference that

↓o and ↑i are chains. (3.6)

So there exists an element d ∈ L\ N5 such that d is neither above the top of this N5, nor below the bottom of this N5. If we suppose i ∥ d; in this case the number of subuniverses is at most 19.75 · 2n−5 by Lemma 2.1(iv) and Lemma 1.3. The case o ∥ d is the same by duality. Since

neither {a, b, d}, nor {c, b, d} is a 3-antichain (3.7)

by (3.3), it follows that d is comparable to a or b and, also, d is compa- rable to c or b. We claim that d ∥ b. Suppose, for a contradiction, that d / ∥ b. (Note, for later reference, that the only assumption on d is that d ∈ L \ (N5 ∪ ↓o ∪ ↑i). By duality, we can assume that d < b. Consider

the element v := a ∨ b. If we had v = i, then {o, i, a, b, c, d} ∼= N ′ would

easily lead to | Sub(L) | ≤ 18.5 · 2n−5 via Lemmas 1.1 and 2.1, whereby v < i. We have that v ≰ b, because otherwise we would obtain that a ≤ b. Since v ≥ b would lead to v = b ∨ v ≥ a ∨ b = i, it follows that v ∥ b. Now if v ≠ c, then we have that

a ∨ b = i, a ∧ b = o, c ∨ b = i, c ∧ b = o, a ∨ d = v, and v ∨ b = i. (3.8)

The seven-element partial lattice {o, i, a, b, c, d, v} defined by these equal- ities has 19.5 · 27−5 subuniverses, whence | Sub(L) ≤ 19.5 · 2n−5 by Lemma 1.3. So, this case cannot occur. On the other hand, if v = c, then the six-element partial lattice {o, i, a, b, c, d} defined by the equal-ities

a ∨ b = i, a ∧ b = o, c ∨ b = i, c ∧ b = o, a ∨ d = i (3.9)

has 21 · 26−5 subuniverses, whence | Sub(L) | ≤ 21 · 2n−5 by Lemma 1.3, and this case is excluded again. Now, we can conclude that d ∥ b. In

fact, taking the assumptions on d into account and using that i ∥ d has

previously been excluded , we have proved that

if x ∈ L \ N5 is not in ↓o ∪ ↑i, then x ∥ b and
o < x < i. (3.10)

Next, armed with d ∥ b, (3.7) implies that {a, c, d} is a chain. There are two subcases depending on d ∈ [a, c] or d ∉ [a, c].

If a < d < c, then {o, i, a, b, c, d} forms a sublattice isomorphic to N6. To ease the notation, we write N6 = {o, i, a, b, c, d}. Using equation 3.2 from [1] , the equality | Sub(N6) | = 21.5 · 26−5 from Lemma 2.1, and Lemma 1.1 (iii), we get that L is not of the form C0 +glu N6 +glu C1 with C0 and C1 being chains. Hence, there is an element e ∈ L \ N6 violating this form. If e ∈ ↓o, then ↓o is not a chain, whence B4 +glu N6 or B4 +glu C(2) +glu N6 is a sublattice of L. But it is straight- forward to compute that | Sub(B4 +glu N6) | = 17.6875 ·2 | B4 +glu N6 | −5 and

| Sub(B + C(2) + N ) | = 17.46875 ·2 | B4 +glu C(2) +glu N6 | −5, whence we can use Lemma 1.1 to exclude that ↓o is not a chain. So, by duality,

both ↓o and ↑i are chains. (3.11)

In particular, e ∉ ↓o ∪ ↑i, and we obtain from (3.10) that e ∥ b. But

{b, d, e} is not a 3-antichain, so we can assume that e < d. No if a < e, then we have a sublattice {o < a < e < d < c < i, o < b < i} such that b is the complement of each of a, e, d, and c. This seven-element sublattice has only 20.75 · 27−5 subuniverses, which excludes the case

a < e in the usual way. So, a ∥ e. Then

b ∧ x = o for every x ∈ {a, e, d, c} and b ∨ y = i for every

$y \in \{a, d, c\}$, (3.12)

and either $a \vee e = d$, or $a \vee e =: u < d$, $u \wedge b = o$, and $u \vee b = i$. At the first alternative, (3.12) together with $a \vee e = d$ defines a seven-element partial sublattices {o, i, a, b, c, d, e} with only $18.5 \cdot 2^{7-5}$ subuniverses, which is excluded in the usual way. At the second alternative, (3.12) together with $a \vee e =: u < d$, $u \wedge b = o$, and $u \vee b = i$ defines an eight-element partial sublattice {o, i, a, b, c, d, e, u} with only $18 \cdot 2^{7-5}$ subuniverses, which is excluded again. We have just excluded that $a < d < c$.

Now that d is not in [a, c], duality allows us to assume that $o < d < a < c$. Let $u := d \vee b$. We can assume that $u < i$, since otherwise $d \vee b = i$ and after interchanging a and d, we are in the previous case. Clearly,

$b \vee d = u$, $b \vee a = i$, $b \vee c = i$, $a \vee u = i$,

$c \vee u = i$, $b \wedge d = o$, $b \wedge a = o$, and $b \wedge c = o$, (3.13)

and these equations define a seven-element partial sublattice {o, i, a, b, c, d, u} with $17.75 \cdot 2^{7-5}$, subuniverses, whereby this case is excluded.

In the case that Θ does collapse two upper edges, We get $H_2$ =

{a, b, c, d, u, i}, where $u := a \wedge b$, see Figure 4. The total number of subuniverses is $19 \cdot 2^{6-5}$ by lemma 2.1(v). Whereby this case is excluded The eight-element partial lattice $H_1$ := {o, 1, a, b, c, d, e, f } see Figure

2 has $19.75 \cdot 2^{8-5}$ subuniverses, by Lemma 2.1(iv), this case is excluded. After excluding all these cases, we have shown the validity of (3.4). To provide a convenient toll to exploit (3.3) and (3.4), we claim that

if $x, y, z \in L$ such that $|\{x, y, z\}| = 3$ and $x \parallel y$, then either $\{x, y\} \subseteq \downarrow z$, or $\{x, y\} \subseteq \uparrow z$, (3.14)

To see this, assume the premise. Since L has no 3-antichain, z is comparable to one of x and y. By duality and symmetry, we can assume that $x < z$. Since $z < y$ would imply $x < y$ and $z \parallel y$ together with $x \parallel y$ would contradict (3.18), we have that $y < z$. This proves (3.14).

Next, by (3.3) and (3.4), we have a four-element subset {a, b, c, d} of

L such that $a \parallel b$ and $c \parallel d$. By duality and (3.14), we can assume that

$\{a, b\} \subseteq \downarrow c$. Applying (3.14) also to {a, b, d}, we obtain that {a, b} is included either in $\uparrow d$, or in $\downarrow d$. Since the first alternative would lead to $d < a < c$ and so it would contradict $c \parallel d$, we have that

$\{a, b\} \subseteq \downarrow d$. Thus, $\{a, b\} \subseteq \downarrow c \cap \downarrow d = \downarrow(c \wedge d)$, and we obtain that $u := a \vee b \leq c \wedge d =: v$. Let $S := \{a \wedge b, a, b, u, v, c, d, c \vee d\}$. Depending on $u = v$ or $u < v$, S is a sublattice isomorphic to $B_4 +_{glu} B_4$ or $B_4 +_{glu} C(2) +_{glu} B_4$. But in the case, it isomorphic to $B_4 +_{glu} B_4$ we have $|Sub(L)| = 21.25 \cdot 2^{n-5}$, which is excluded in our claim. Using Lemma 1.1 together with (i) and (xi) of Lemma 2.1, we obtain that

$|Sub(L)| \leq 21.125 \cdot 2^{n-5}$ and $|Sub(L)| = 21.125 \cdot 2^{n-5}$ holds only when

L is of form (3.1).

We prove part (ii).

Let L be an n-element lattice. We obtain from Lemma 1.1 (iii) and from 2.1(vii) that if

$L \cong C_0 +_{glu} N_7 +_{glu} C_1$ for finite chains $C_0$ and $C_1$, (3.15)

then $|Sub(L)| = 20.75 \cdot 2^{n-5}$. In order to complete the proof of The- orem 3.1 (ii), it suffices to exclude the existence of a lattice L such that

$|L| = n$, $20.75 \cdot 2^{n-5} \leq |Sub(L)| < 21.125 \cdot 2^{n-5}$, but L is not of the form given in (3.15). (3.16)

Suppose, for a contradiction, that L is a lattice satisfying (3.16). Then, by Theorem 1.1 of [6] and Lemma 3.3 of [6],

L has at least two 2-antichains but it has no 3-antichain. (3.17)

We prove that

L cannot have two non-disjoint 2-antichains. (3.18)

Suppose to the contrary that {a, b} and {c, b} are two distinct 2- antichains in L. Since there is no 3-antichain in L, we can assume that $a < c$. With $K := [\{a, b, c\}]$, let $\varphi : Flat(\tilde{a}, \tilde{b}, \tilde{c}) \to K$ be the unique lattice homomorphism from Lemma 3.3, and let Θ be the kernel of φ.

We follow the notations of Figure 6.

First, we investigate the case when Θ does not collapse $e_1$ and at least one of $e_4$ or $e_6$. By duality, we can assume that $e_4$ is not collapsed. Since $e_1$ generates the monolith congruence, i.e., the smallest nontrivial congruence of the $N_5$ sublattice of Flat $(\tilde{a}, \tilde{b}, \tilde{c})$, no other edge of the $N_5$ sublattice is collapsed. Now, $e_4$ is perspective to $e_5$, $e_9$ is perspective to

$e_8$. Hence, $N_5B_4$ is a sublattice of L and we conclude that $|Sub(L)| \leq$

$17.25 \cdot 2^{n-5}$ by Lemma 1.1 (ii) and by Lemma 2.1 (ii).

So, if Θ does not collapse $e_1$, then it collapses both $e_4$ and $e_6$. Since in this case $e_1$ also generates the monolith congruence of the $N_5$ sublattice of $Flat(\tilde{a}, \tilde{b}, \tilde{c})$, no other edge of this $N_5$ sublattice is collapsed. Hence

{a, b, c} generates a pentagon sublattice $N_5$ of L. We know from [6] that

$|Sub(N_5)| = 23$, and we also have assumed in (3.16) that $|Sub(L)| < 23 \cdot 2^{n-5}$. Thus, it follows from Lemma 1.1 (iii) that L cannot be of form (3.16) Similarly to case (i), again, if $\downarrow o$ is not a chain, then we would have a sublattice of form either $B_4 +_{glu} B_4$ or $B_4 +_{glu} C_1 +_{glu} B_4$, but then the number of subuniverses could be at most $21.25 \cdot 2^{n-5}$. By Lemma 1.1 (ii), moreover $21.25 \cdot 2^{n-5}$ can appear only in case that L is of form $C_0 +_{glu} B_4 +_{glu} B_4 +_{glu} C_1$ as proved in [1] . Hence, $\downarrow o$ is a chain. We obtain, by duality, for later reference that

$\downarrow o$ and $\uparrow i$ are chains. (3.19)

The situation that there exists an element d ∈ [o, i] \ N5 together with the absence of 3-antichains imply that d must be comparable either with b or with a and c. But then L has either N6 or N ′ as a sublattice and Lemma 1.1 and Lemma 2.1(ix) and (x) yields that L has either at most 21.5 ·2n−5 or at most 18.5 ·2n−5 sublattices. In case L has N6 sublattice, by Lemma 1.1 (iii), 21.5 ·2n−5 but this has been excluded in (3.16). By duality, we are left with the case when there exists an element d ∈ L \ N5 such that d is neither above i nor below o and i ∥ d then the number of subuniverses is at most 19.75 ·2n−5 by Lemma 2.1(iv) and Lemma 1.3.

The case Θ does collapse e2 or e3, Then N6B4 is sublattice of L and

we conclude that | Sub(L)| ≤ 16.5 · 2n−5 by lemma 2.1(viii) that case also excluded.

Second, we investigate the case when Θ does collapse e1. Since a ∥ b and c ∥ b, none of the thick edges e8, . . . , e11 is collapsed by Θ. Observe that at least one of e4 and e6 is not collapsed by Θ, since otherwise

⟨a˜, c˜⟩ would belong to Θ = ker(φ) by transitivity and a = c would be a contradiction. By duality, we can assume that e4 is not collapsed by Θ. Since e2, e3, and e5 are perspective to e10, e9, and e4, respectively, these edges are not collapsed either. So, with the exception of e1, no edge among the elements denoted by big circles in Figure 6 is collapsed. Thus, the φ-images of the "big" elements form a sublattice (isomorphic to) C(2) × C(3) in L. Hence, | Sub(L)| ≤ 19 ·2n−5 by Lemma 1.1 (ii) and 2.1(iii), which contradicts our assumption that L satisfies (3.16). This proves (3.18).

Similarly, to (3.14), the same claim here also holds (because of (3.17) and (3.18)), namely

if x, y, z ∈ L such that |{x, y, z}| = 3 and x ∥ y, then either {x, y} ⊆ ↓z, or {x, y} ⊆ ↑z, (3.20)

and its proof is also the same.

Next, by (3.17) and (3.18), we have a four-element subset {a, b, c, d} of L such that a ∥ b and c ∥ d. By duality and (3.20), we can assume that {a, b} ⊆ ↓c. Applying (3.20) also to {a, b, d}, we obtain that

{a, b} is included either in ↑d, or in ↓d. Since the first alternative would lead to d < a < c and so it would contradict c ∥ d, we have that

{a, b} ⊆ ↓d. Thus, {a, b} ⊆ ↓c ∩ ↓d = ↓(c ∧ d), and we obtain that u := a ∨ b ≤ c ∧ d =: v. Let S := {a ∧ b, a, b, u, v, c, d, c ∨ d}. Depending on u = v or u < v, S is a sublattice isomorphic to B4 +glu B4 or B4 +glu C(2) +glu B4. Using Lemma 1.2 of together with (i) and (ix) of Lemma 2.1, we obtain that | Sub(L)| ≤ 21.25 · 2n−5. This inequality contradicts (3.16) and completes the proof of part(ii) of Theorem 3.1.

## 4. CONCLUSION

We proved that the sixth largest number of subuniverses of an n- element lattice is 21.125 · 2n−5 and the seventh largest number is 20.75 · 2n−5. Also, we described the n-element lattices with exactly 21.125 ·2n−5

which is L ∼= C0 +glu B4 +glu C2 +glu B4 +glu C1 and 20.75 ·2n−5 subuni-

verses which is L ∼= C0 +glu N7 +glu C1. One of the most important

applications of this result is in road systems with traffic lights.

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