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Existence and Uniqueness Solution of Certain Integral Equation

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ABSTRACT

The aim of this work is to study the existence and uniqueness solution of certain integral equation by using Picard approximation method and Banach fixed point theorem. The study of integral equation is more general and leads us to improve and extend some results of Butris.

KEY WORDS: Picard Approximation method, Banach fixed point theorem, Integral Equation.

1. INTRODUCTION

An Integral equation is an equation in which the unknown function $u(x)$ to be determined appears under the integral sign. A typical form of an integral equation in $u(x)$ is of the form

$$u(x) = f(x) + \int_{\alpha(x)}^{\beta(x)} K(x,t)u(t)dt,$$

where $K(x,t)$ is called the kernel of the integral equation, and $\alpha(x)$ and $\beta(x)$ are the limits of integration.

Integral equations arise naturally in physics, chemistry, biology and engineering applications modeled by initial value problems for finite interval $[a,b]$. The study solutions of integral equations have been use in many problems for example [1,2,4,5,6,7,8].

Integral equation has been arisen in many mathematical and engineering field, so that solving this kind of problems are more efficient and useful in many research branches. Analytical solution of this kind of equation is not accessible in general form of equation and we can only get an exact solution only in special cases. But in industrial problems we have not spatial cases so that we try to solve this kind of equations numerically in general format. Many numerical schemes are employed to give an approximate solution with sufficient accuracy [3, 12, 14, 15, 17,18].

Integral equations of various types and kinds play an important role in many branches of mathematics. Over the past thirty years substantial progress has been made in developing innovative approximate analytical for example see[8,11,13,15,16,17].

Definition 1. [1].

Let $\{z_m(t)\}_{m=0}^{\infty}$ be a sequence of functions defined on a set $E \subseteq R$. We say that $\{z_m(t)\}_{m=0}^{\infty}$ converges uniformly to the limit function z on E if, givenδ, $\varepsilon(z_{0,}\delta) > 0$, such that:

$$|z_m(t) - z(t)| < \varepsilon \quad , \qquad t \in E \quad .$$

Definition 2. [1].

Let z be a continuous function defined on a domain $D = \{(t,z): a \le t \le b, c \le z \le d\}$. Then z is said to satisfy a Lipschitz condition in the variable x on G, provided that a constant $L > 0$ exists with the property that $|z(t,z_1) - z(t,z_2)| \le L|z_1 - z_2|$, for all $(t,z_1),(t,z_2) \in D$. The constant L is called a Lipchitz constant for z .

Theorem 1. [1]. If a sequence of continuous function $\{z_m(t)\}_{m=0}^{\infty}$ defined on a common interval $I \in D$ converges uniformly on I to the limit function z, then z is continuous on I .

Butris [1] used Picard approximation method and Banach fixed point theorem for studying the existence and uniqueness solutions for a system of integral equation which has the following form

$$u(x) = f(x) + \int_a^x F(x,y)u(y)dy ,$$

where $x \in G \subseteq R^n$, G is a closed and bounded domain.

Our work is considered by the following Integral equation: -

$$z(t) = z_0 e^{At} + \int_0^t e^{A(t-s)} g(s,z(s))\gamma(t,s)ds + h(t) \ldots \quad (1)$$

where $t \in I_a = [0,a], s \in I_b = [0,b]$.

Suppose that $g(t,z(t))$ and $h(t)$ are defined, continuous on the domain

$$D = \{(t,z): \ 0 \le t \le a, 0 \le z \le b\} \qquad \ldots \quad (2)$$

Also, the vector functions $g(t,z(t))$ and $h(t)$ are satisfying the following inequalities:

$$\|g(t,z)\| \le M \qquad \ldots \quad (3)$$

$$\|g(t,z_1) - g(t,z_2)\| \le K\|z_1 - z_2\| \qquad \ldots \quad (4)$$

$$\|h(t)\| \le N \qquad \ldots \quad (5)$$

With the singular kernel $\gamma(t,s)$ sush that

$\|\gamma(t,s)\| = \int_0^{h(t)} \|H(t,s)\|ds$,where $\|H(t,s)\| \le M$ … (6)

and

$$\left\|e^{A(t-s)}\right\| \le Q \qquad \ldots \quad (7)$$

where M, K, N, H and Q are positive constants.

Define a sequence of vector functions $\{z_m(t)\}_{m=0}^{\infty}$ by

$$z_{m+1}(t) = z_0 e^{At} + \int_0^t e^{A(t-s)} g(s,z_m(s))\gamma(t,s)ds + h(t) \qquad \ldots \quad (8)$$

with

$z_0(t) = z_0 e^{At}$, $m = 0,1,2,3,\ldots, t \in I_a, s \in I_b$.

Also define a non-empty set as follows:

$$D_Z = D - (QMHNa + N) \qquad \ldots \quad (9)$$

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2. PROCEDURE FOR PAPER SUBMISSION

In this section, we study the existence and uniqueness of integral equation (1) by using Picard approximation method which are given by[2].

Theorem1.

Let the vector functions $g(t,z(s)), \gamma(t,s)$ and $h(t)$ are defined and continuous on the domain (2). Suppose that the vector functions $g(t,z(s))$ and $\gamma(t,s)$ satisfying the inequalities (3), (4), (5), (6), (7) and the relation (9). Then there exists a sequence of functions (8) converges uniformly to the limit functions $z = z(t)$ which is defined by following integral equation:-

$$z(t) = z_0 e^{At} + \int_0^t e^{A(t-s)} g(t, z(s))\gamma(t,s)ds + h(t) \quad \dots (10)$$

Is continuous on the same domain, provided that

$\|z_m(t) - z_0\| \leq QMHNa + N \quad \dots \quad (11)$

and

$$\|z_{m+1}(t) - z_m(t)\| \leq (QKHNa)^m \, \|z_1(t) - z_0\| \quad \dots \quad (12)$$

for all m=0,1,2,… .

We shall prove the following steps

I. $\{z_m(t)\}_{m=0}^{\infty} \in D$, for all $t \in I_a$,

II. $\lim_{m\to\infty} z_m(t) = z(t)$,(converges uniformly),

III. $z(t) \epsilon D$, for all $t \in I_a$,

IV. $z(t)$ is a unique solution of (1).

Proof(i). For $m = 1$ in (8), we have

$$\|z_1(t) - z_0\| \leq \int_0^t \left\|e^{A(t-s)}\right\| \|g(s, z_0)\| \|\gamma(t,s)\| ds + \|h(t)\|$$

$$\leq Q \, \|g(t, z_0)\| \left[\int_0^{h(t)} \|H(t,s)\| ds\right] \int_0^t ds + N$$

$\leq QMH\|h(t)\|a + N$

So that

$\|z_1(t) - z_0\| \leq QMHNa + N$

i.e $z_1(t) \in D$, for all $t \in I_a$.

For $m = 2$ in (8), we have

$$\|z_2(t) - z_0\| \leq \int_0^t \left\|e^{A(t-s)}\right\| \|g(s, z_1)\| \|\gamma(t,s)\| ds + \|h(t)\|$$

$$\leq Q \, \|g(t, z_1(t))\| \left[\int_0^{h(t)} \|H(t,s)\| ds\right] \int_0^t ds + N$$

$\leq QMH\|h(t)\|a + N$

thus

$\|z_2(t) - z_0\| \leq QMHNa + N$

i.e $z_2(t) \in D$, for all $t \in I_a$.

By mathematical induction, we can prove that $z_m(t) \in C(D)$, for all $t \in I_a$, $m = 1,2,3, \dots$.

Proof(ii). For $m = 1$ in (8), we have

$$\|z_2(t) - z_1(t)\| \leq \int_0^t \left\|e^{A(t-s)}\right\| \|g(s, z_1(s)) - g(s, z_0)\| \|\gamma(t,s)\| ds$$

$$\leq \int_0^t QK \, \|z_1(s) - z_0\| \left[\int_0^{h(t)} H \, ds\right] ds$$

$\leq QK\|z_1(t) - z_0\|HNa$

hence

$\|z_2(t) - z_1(t)\| \leq QKHNa\|z_1(t) - z_0\|$

For $m = 2$ in (8), we have

$$\|z_3(t) - z_2\| = \left\| z_0 e^{At} + \int_0^t e^{A(t-s)} g(s, z_2(s))\gamma(t,s)ds + h(t) - z_0 e^{At} - \int_0^t e^{A(t-s)} g(s, z_1)\gamma(t,s)ds - h(t) \right\|$$

$$\leq \int_0^t \left\|e^{A(t-s)}\right\| \|g(s,z_2(s)) - g(s,z_1(s))\| \|\gamma(t,s)\| ds$$
$$\leq \int_0^t QK \, \|z_2(s) - z_1(s)\| \left[\int_0^{h(t)} H \, ds\right] d$$
$$\leq KQ\|z_2(t) - z_1\|HNa$$
so
$$\|z_3(t) - z_2(t)\| \leq (QKHNa)^2 \, \|z_1(t) - z_0\|$$

And so on, by mathematical induction, we have
$$\|z_{m+1}(t) - z_m(t)\| \leq (QKHNa)^m \, \|z_1(t) - z_0\|$$
Suppose that $L = QKHNa < 1$, we have
$$\sum_{i=0}^{k} \|z_{m+i}(t) - z_m(t)\| \leq (1 + L + L^2 + \cdots + L^m + \cdots) \, L \, \|z_1(t) - z_0\|$$
$$\leq \frac{L}{1-L} \|z_1(t) - z_0\|$$
Therefore, the sequence of functions $\{z_m(t)\}_{m=0}^{\infty}$ converges uniformly on the domain D.

Proof (iii). We shall prove that $z(t) \in D$, i.e.
$$\lim_{m \to \infty} \int_0^t e^{A(t-s)} g(t, z_m(s)) \gamma(t,s) ds + h(t) = \int_0^t e^{A(t-s)} g(s, z(s)) \gamma(t,s) ds + h(t)$$
and

Taking
$$\left\| z_0 e^{At} + \int_0^t e^{A(t-s)} g(t, z_m(s)) \gamma(t,s) ds + h(t) - z_0 e^{At} - \int_0^t e^{A(t-s)} \gamma(t,s) ds - h(t) \right\|$$
$$\leq \int_0^t \left\|e^{A(t-s)}\right\| \|g(s,z_m(s)) - g(s,z\,(s))\| \|\gamma(t,s)\| ds$$
$$\leq \int_0^t QK \, \|z_m(s) - z(s)\,\| \left[\int_0^{h(t)} H \, ds\right] ds$$
$$\leq QKHNa\|z_m(t) - z\,(t)\|$$
Since $\{z_m(t)\}_{m=0}^{\infty}$ is convergent uniformly, then $\lim_{m \to \infty} z_m(t) = z(t)$. i.e. $\|z_m(t) - z\,(t)\| \leq \epsilon_1$. Choosing $\epsilon_1 = \frac{\epsilon}{QKHNa}$, we get
$$\|z_m(t) - z\,(t)\| \leq \epsilon$$
and hence $z(t) \in D,\ t \in I_a$.

Proof (iv). Suppose that
$$\bar{z}(t) = z_0 e^{At} + \int_0^t e^{A(t-s)} g(s, \bar{z}(s)) \gamma(t,s) ds + h(t)$$
Is another solution of (1).
For $m = 1$ in (8), we have

$$\|z_1(t) - \bar{z}\,(t)\| = \left\| z_0 e^{At} + \int_0^t e^{A(t-s)} g(s, z_0) \gamma(t,s) ds + h(t) - z_0 e^{At} - \int_0^t e^{A(t-s)} g(s, \bar{z}\,(s)) \gamma(t,s) ds - h(t) \right\|$$
$$\leq \int_0^t \left\|e^{A(t-s)}\right\| \|g(s,z_0) - g(s,\bar{z}\,(s))\| \|\gamma(t,s)\| ds$$

$$\leq \int_0^t QK\, \|z_0 - \bar{z}(s)\| \left[\int_0^{h(t)} H\, ds \right] ds$$
$\leq QK\|z_0 - \bar{z}(t)\|HNa$
$\leq QKHNa\, \|\bar{z}(t) - z_0\|$
For $m = 2$ in (8), we have

$$\|z_2(t) - \bar{z}\,(t)\| = \left\| z_0 e^{At} + \int_0^t e^{A(t-s)} g(s, z_1(s)\gamma(t,s)ds + h(t) - z_0 e^{At} - \int_0^t e^{A(t-s)} g(s, \bar{z}\,(s))\gamma(t,s)ds - h(t) \right\|$$

$$\leq \int_0^t \|e^{A(t-s)}\| \|g(s, z_1(s)) - g(s, \bar{z}\,(s))\| \|\gamma(t,s)\| ds$$

$$\leq \int_0^t QK\, \|z_1(s) - \bar{z}(s)\| \left[\int_0^{h(t)} H\, ds \right] ds$$
$\leq QK\|z_1(t) - \bar{z}(t)\|HNa$
$\leq QKHNa\, \|\bar{z}(t) - z_1(t)\|$
$\leq (QKHNa)^2\, \|\bar{z}(t) - z_0\|$
By mathematical induction, we have

$$\|z_m(t) - z\,(t)\| \leq \frac{L}{1-L} \|\bar{z}(t) - z_0\|$$
Since we have $L < 1$, then

$$\lim_{m\to\infty} z_m(t) = \bar{z}(t) = z(t).$$
Then $\bar{z}(t) = z(t)$. Hence $z(t)$ is a unique solution of (1) on D. ■

3. Banach fixed point theorem.

In this section ,we study the existence and uniqueness solution of certain integral equation (1) by using Banach fixed point theorem which are given by[1].

Theorem 2. Let $g\big(t, z(t)\big) \in C(D), h(t) \in C(D)$. Then the integral equation (1) has a unique continuous solution $z(t)$ on I_a satisfying the condition $z(0) = z_0 e^{At}$.

Proof. Let $(D, \|\cdot\|)$ be Banach space.

Define a mapping T^* on S as follows:-

$$T^* z(t) = z_0 e^{At} + \int_0^t e^{A(t-s)} g(s, z\,(s))\gamma(t,s)ds + h(t)$$
where $(s \in I_a, t \in I_b)$. Then we shall prove that

I. $T^*: D \to D$, for all $z(t) \in D \Longrightarrow T^* z(t) \in D$.

II. T^* is a contraction mapping on D.

Since $g(t, z(t))$ is continuous on the domain (2) and $h(t) \in D$, then $\int_0^t e^{A(t-s)} g(s, z\,(s))\gamma(t,s)ds$ is also continuous on the same domain.

Thus, we get

$$z_0 e^{At} + \int_0^t e^{A(t-s)} g(s, z\,(s))\gamma(t,s)ds + h(t)$$
which is continuous in the domain (2). i.e. $T^*: D \to D$.

Now, we claim that T^* is a contraction mapping on S.

Let $z(t)$ and $\bar{z}(t) \in D$, then

$$\|T^* z\,(t) - T^* \bar{z}\,(t)\| = \left\| z_0 e^{At} + \int_0^t e^{A(t-s)} g(s, z\,(s))\gamma(t,s)ds + h(t) - z_0 e^{At} - \int_0^t e^{A(t-s)} g(s, \bar{z}\,(s))\gamma(t,s)ds - h(t) \right\|$$

$$\leq \int_0^t \|e^{A(t-s)}\| \|g(s, z(s)) - g(s, \bar{z}(s))\| \|\gamma(t,s)\| ds$$

$$\leq \int_0^t QK \, \|z(s) - \bar{z}(s)\| \left[\int_0^{h(t)} H \, ds \right] ds$$

$\leq QK\|z(t) - \bar{z}(t)\| HNa$

therefor

$\|T^* z(t) - T^* \bar{z}(t)\| \leq QKHNa \, \|z(t) - \bar{z}(t)\|$

Since $QKHNa < 1$, thus T^* is a contraction mapping on S. That is

$\|T^* z(t) - T^* \bar{z}(t)\| \leq \|z(t) - \bar{z}(t)\|$

Then, T^* has a unique fixed point $z \in D$. i.e. $T^* z(t) = z(t)$ and

$$z(t) = z_0 e^{At} + \int_0^t e^{A(t-s)} g(s, z(s)) \gamma(t,s) ds + h(t)$$

Hence $z(t)$ is a unique continuous solution of the Volterra integral equation (1) satisfying the condition$z(0) = z_0 e^{At}$.

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