Some numerical graphs with successive approximation method of fractional integro–differential equation

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ABSTRACT

This paper studies the existence and uniqueness solution of fractional integro-differential equation, by using some numerical graphs with successive approximation method of fractional integro –differential equation. The results of written new program in Mat-Lab show that the method is very interested and efficient. Also we extend the results of Butris [3].

Keywords: fractional Integro-differential equations, Successive approximations method, Mat-lab program method, Existence and Uniqueness,

1. Introduction

Fractional differential equations have been of great interest recently. It is caused both by the intensive development of theory of fractional calculus itself and by application of such constructions in various sciences such as physics, chemistry, mechanics, engineering, For details, see [4, 5, 6, 7,8]. Burris[3]has study a solution of integro-differential equation of fractional order which has the form:

$$x^{(\alpha)}(t) = f\left(t, x, \int_{t-T}^{t} g\big(sx(s)\big)ds\right), x_0 \quad , \ 0 < \alpha < 1$$

where $x \in D_\alpha \subseteq [0,T]$ and D_α is a closed and bounded domain.

This paper use some numerical graphs with successive approximation method of fractional integro–differential equation and also extend the results of Butris [3],where D^α is the standard Riemann – Liouville fractional derivative.

Definition

Let f be a function which is defined a. e. (almost every where) on $[a,b]$. For $\alpha > 0$, we define:

$$\underset{a}{\overset{b}{\mathrm{I}}}{}^{\alpha} f = \frac{1}{\Gamma(\alpha)} \int_a^b (b-s)^{\alpha-1} f(s)ds$$

Provided that this integral (Lebesgue) exists.

Definition 2.[2]. If $\alpha > 0$, then Gamma's function is denote by (Γ) and defined by the form:

$$\Gamma(\alpha) = \int_0^\infty e^{-s} s^{\alpha-1} ds$$

Lemma 1[1].

If $\{f_n\}_{n=1}^{\infty}$ is a sequences of functions is defined on the set $E \subseteq R$ such that $|f_n| \le M_n$, where M_n is a positive number, then $\sum_{n=1}^{\infty} f_n$ is

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uniformaly convergent on E if $\sum_{n=1}^{\infty} M_n$ is convergent.

Lemma 2[1].

Let $E_\alpha(m;x)=\sum_{m=1}^{\infty}\frac{m^{n-1}x^{n\alpha-1}}{\Gamma(n\alpha)}$, where m = R , then:

- the series converges for $x \neq 0$ and $\alpha > 0$.
- the series converges everywhere when $\alpha \geq 0$.
- if $\alpha = 0$, then $E_1(m,x)=\exp(mx)$.

Lemma 3[3].

If K_1 and K_2 be a positive constant, and f be a continuous function on $a \le t \le b$, such that:

$$f(t) \le K_1 + K_2 \int_a^t f(s)ds$$

Then

$$f(t) \le K_1 \exp\left(K_2(t-a)\right)$$

This paper deals some numerical graphs with successive approximation method of fractional integro –differential equation which has the form:-

$$x^{(\alpha)}(t) = f\left(t, x(t), h(t)x(t), \int_{t-T}^{t} p\big(s, x(s), h(s)x(s)\big)ds\right) \quad (1)$$

$$x^{(\alpha-1)} = x_0\ ,\ \ 0<\alpha<1$$

The function $f\big(t, x(t), h(t)x(t), g(t)\big)$ is defined, continuous on the domain:

$$(t,x) \in [0,T]\times D_\alpha \quad (2)$$

where $x \in D_\alpha \subseteq [0,T]$, D_α is a closed and bounded domain subset of R.

We denote to $\int_{t-T}^{t} P\big(t, x(t), h(t)x(t)\big)dt$ by $g(t)$,

Suppose that the functions $f\big(t, x(t), h(t)x(t), g(t)\big)$, $h(t)$ satisfies the following inequalities:

$$\left\| f\big(t, x(t), h(t)x(t), g(t)\big)\right\| \le M \quad (3)$$

$$\left\| f\big(t,x_1,hx_1,g_1(t)\big) - f\big(t,x_2,hx_2,g_2(t)\big)\right\| \le L\|x_1-x_2\| + NL\|x_1-x_2\| + K\big(L\|x_1-x_2\| + NL\|x_1-x_2\|\big) \quad \textbf{(4)}$$

for all $t\in[0,T]$, and $x, x_1, x_2 \in D_\alpha$, where L, M, N, K , are positive constants, $\|.\| = \max|.|$ .Here h(t) is a continuous function in t provided that:

$$\|h(t)\| \le N\ ,\ \mathbf{N>0} \quad \textbf{(5)}$$

We define the non-empty sets as follows:

$$D_{\alpha f} = D_\alpha - \frac{T^\alpha}{\Gamma(\alpha+1)}M\ ,\ D_{\alpha f} \ne \varphi \quad \textbf{(6)}$$

Moreover, we suppose the value of the following equation:

$$\omega = \frac{T^\alpha}{\Gamma(\alpha+1)}\eta < 1 \quad \textbf{(7)}$$

2. Existenece of Solution

Theorem 1.

Let the vector function $f\big(t, x(t), h(t)x(t), Q(t)\big)$ be defined in the domain (2), continuous in t, x and satisfy the inequalities (3),(4), and (5) , then the function:

$$x(t,x_0) = x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t\left[f\left(s, x(s,x_0), h(s)x(s,x_0), \int_{s-T}^{s} P\big(\tau, x(\tau,x_0), h(\tau)x(\tau,x_0)\big)d\tau\right)\right](t-s)^{\alpha-1}ds$$

is a solution of (1).

Proof:

Let

$$x_{m+1}(t,x_0) = x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t\left[f\left(s, x_m(s,x_0), h(s)x_m(s,x_0), \int_{s-T}^{s} P\big(\tau, x_m(\tau,x_0), h(\tau)x_m(\tau,x_0)\big)d\tau\right)\right](t-s)^{\alpha-1}ds$$

(8)

With $\boldsymbol{x_0^{(\alpha-1)}(t,x_0) = x_0}$ $\quad \boldsymbol{m = 0,1,2,\ldots}$

be a sequence of functions which is defined on the domain:

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$$(t,x_0)\in[0,T]\times D_{\alpha f} \tag{9}$$

We will divide the proof as follows:

(i) $x_m(t,x_0)\in D_\alpha$ **, for all** $t\in[0,T]$**,** $x_0\in D_{\alpha f}$**.**

(ii) $x_m(t,x_0)\in D_\alpha$, is uniformly convergent to the function $x(t,x_0)$ on the domain(9) , for all $t\in[0,T]$, $x_0\in D_{\alpha f}$

(iii) $x(t,x_0)\in D_\alpha$, for all $t\in[0,T]$, $x_0\in D_{\alpha f}$.

proof (i):

Set m=0 and use (8), we get:

$$\|x_1(t,x_0)-x_0\| = \left\| x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t \left[f\left(s, x_1(s,x_0), h(s)x_1(s,x_0), \int_{s-T}^{s} P(\tau, x_1(\tau,x_0), h(\tau)x_1(\tau,x_0))d\tau\right)\right](t-s)^{\alpha-1}ds - x_0\right\|$$

$$\le \frac{1}{\Gamma(\alpha)}\int_0^t \left\| f\left(s, x_1(s,x_0), h(s)x_1(s,x_0), \int_{s-T}^{s} P(\tau, x_1(\tau,x_0), h(\tau)x_1(\tau,x_0))d\tau\right)\right\|(t-s)^{\alpha-1}ds$$

$$\le \frac{1}{\Gamma(\alpha)}\int_0^t M(t-s)^{\alpha-1}ds$$

$$\le \frac{t^\alpha}{\Gamma(\alpha+1)}M \quad , \quad t\in[0,T]$$

$$\|x_1(t,x_0)-x_0\| \le \frac{T^\alpha}{\Gamma(\alpha+1)}M \tag{10}$$

That is $x_1(t,x_0)\in D_\alpha$, for all $t\in[0,T]$, $x_0\in D_{\alpha f}$.

By mathematical induction we have:

$$\|x_m(t,x_0)-x_0\| = \left\| x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t \left[f\left(s, x_m(s,x_0), h(s)x_m(s,x_0), \int_{s-T}^{s} P(\tau, x_m(\tau,x_0), h(\tau)x_m(\tau,x_0))d\tau\right)\right](t-s)^{\alpha-1}ds - x_0\right\|$$

$$\|x_m(t,x_0)-x_0\| \le \frac{T^\alpha}{\Gamma(\alpha+1)}M$$

where. $x_0\in D_{\alpha f}$, $t\in[0,T]$, for all $x_m(t,x_0)\in D_\alpha$

proof (ii):

Now, we shall prove that the sequence of functions (9) is uniformly convergent on (9). From (8), when m=1 we get:

$$\|x_2(t,x_0)-x_1(t,x_0)\| = \left\| x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t \left[f\left(s, x_1(s,x_0), h(s)x_1(s,x_0), \int_{s-T}^{s} P(\tau, x_1(\tau,x_0), h(\tau)x_1(\tau,x_0))d\tau\right)\right](t-s)^{\alpha-1}ds - \right.$$

$$\left. - x_0 - \frac{1}{\Gamma(\alpha)}\int_0^t \left[f\left(s, x_0(s,x_0), h(s)x_0(s,x_0), \int_{s-T}^{s} P(\tau, x_0(\tau,x_0), h(\tau)x_0(\tau,x_0))d\tau\right)\right](t-s)^{\alpha-1}ds \right\|$$

$$\le \frac{1}{\Gamma(\alpha)}\int_0^t \left[L\|x_1(\tau,x_0)-x_0(\tau,x_0)\| + NL\|x_1(\tau,x_0)-x_0(\tau,x_0)\| + \int_{s-T}^{s}\left(L\|x_1(\tau,x_0)-x_0(\tau,x_0)\| + NL\|x_1(\tau,x_0)-x_0(\tau,x_0)\|\right)d\tau \right](t-s)^{\alpha-1}ds$$

$$\le \frac{1}{\Gamma(\alpha)}\int_0^t \left[(1+N)L\|x_1(\tau,x_0)-x_0(\tau,x_0)\| - (1+N)L\|x_1(\tau,x_0)-x_0(\tau,x_0)\|\right](t-s)^{\alpha-1}ds$$

$$\le \frac{1}{\Gamma(\alpha)}(1+N)\int_0^t \left[L\|x_1(\tau,x_0)-x_0(\tau,x_0)\|\right](t-s)^{\alpha-1}ds$$

$$\le \frac{1}{\Gamma(\alpha)}(1+N)\int_0^t \left[L\frac{T^\alpha}{\Gamma(\alpha+1)}M\right](t-s)^{\alpha-1}ds$$

$$\le \eta\omega^2 M$$

and hence

$$\|x_2(t,x_0)-x_1(t,x_0)\| \le \omega^2 M$$

Now when m=2 in (8) we get:

$$\|x_3(t,x_0)-x_2(t,x_0)\| = \left\| x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t \left[f\left(s, x_2(s,x_0), h(s)x_2(s,x_0), \int_{s-T}^{s} P(\tau, x_2(\tau,x_0), h(\tau)x_2(\tau,x_0))d\tau\right)\right](t-s)^{\alpha-1}ds - \right.$$

$$\left. - x_0 - \frac{1}{\Gamma(\alpha)}\int_0^t \left[f\left(s, x_1(s,x_0), h(s)x_1(s,x_0), \int_{s-T}^{s} P(\tau, x_1(\tau,x_0), h(\tau)x_1(\tau,x_0))d\tau\right)\right](t-s)^{\alpha-1}ds \right\|$$

$$\le \frac{1}{\Gamma(\alpha)}\int_0^t \left[L\|x_2(\tau,x_0)-x_1(\tau,x_0)\| + NL\|x_2(\tau,x_0)-x_1(\tau,x_0)\| + \right.$$

$$\left. + \int_{-\infty}^{s} \delta e^{-\lambda(t-s)}\left(L\|x_2(\tau,x_0)-x_1(\tau,x_0)\| + NL\|x_2(\tau,x_0)-x_1(\tau,x_0)\|\right)d\tau \right](t-s)^{\alpha-1}ds$$

$$\leq \frac{1}{\Gamma(\alpha)}\int_0^t \left[(1+N)L\left\|x_2(\tau,x_0)-x_1(\tau,x_0)\right\|-(1+N)L\left\|x_2(\tau,x_0)-x_1(\tau,x_0)\right\|\right](t-s)^{\alpha-1}ds$$

$$\leq \frac{1}{\Gamma(\alpha)}(1+N)\int_0^t\left[L\left(\frac{T^\alpha}{\Gamma(\alpha+1)}\right)^2 M\right](t-s)^{\alpha-1}ds$$

$\leq \eta^2\omega^3 M$

Therefore

$\|\mathbf{x_3(t,x_0)-x_2(t,x_0)}\| \leq \eta^2\boldsymbol{\omega^3}\mathbf{M}$

Then by mathematical induction we have:

$\|\mathbf{x_{m+1}(t,x_0)-x_m(t,x_0)}\| \leq \eta^m\boldsymbol{\omega^{m+1}}\mathbf{M}$ **(11)**

For all m=0,1,2,.... .

Now from (11), and for $p \geq 1$, we get:

$$\left\|x_{m+p}(t,x_0)-x_m(t,x_0)\right\| \leq M\sum_{i=0}^{p-1}\omega^{i+1}\eta^i \quad (12)$$

WHERE

$$\left\|x_{m+p}(t,x_0)-x_m(t,x_0)\right\| = \left\|x_{m+p}(t,x_0)-x_{m+p-1}(t,x_0)\right\| +$$

$$+\left\|x_{m+p-1}(t,x_0)-x_{m+p-2}(t,x_0)\right\| + \ldots\ldots\ldots\ldots + \left\|x_{m+1}(t,x_0)-x_m(t,x_0)\right\|$$

$$\leq \left(\frac{T^\alpha}{\Gamma(\alpha+1)}\eta\right)^{m+p-1}\left\|x_1(t,x_0)-x_0\right\| + \left(\frac{T^\alpha}{\Gamma(\alpha+1)}\eta\right)^{m+p-2}\left\|x_1(t,x_0)-x_0\right\| + \ldots\ldots + \left(\frac{T^\alpha}{\Gamma(\alpha+1)}\eta\right)^{m}\left\|x_1(t,x_0)-x_0\right\|$$

SO THAT

$$\left\|x_{m+p}(t,x_0)-x_m(t,x_0)\right\| \leq \left(\frac{T^\alpha}{\Gamma(\alpha+1)}\eta\right)^m\left[1+\left(\frac{T^\alpha}{\Gamma(\alpha+1)}\eta\right)+\left(\frac{T^\alpha}{\Gamma(\alpha+1)}\eta\right)^2+\ldots+\left(\frac{T^\alpha}{\Gamma(\alpha+1)}\eta\right)^{p-2}+\left(\frac{T^\alpha}{\Gamma(\alpha+1)}\eta\right)^{p-1}\right]\left\|x_1(t,x_0)-x_0\right\| \quad \textbf{(13)}$$

We note that the right hand from (13) is bounded with the convergent geometric series and its summation to equals $\frac{1}{1-\Psi}$ and hence

$$\left\|x_{m+p}(t,x_0)-x_m(t,x_0)\right\| \leq \left(\frac{T^\alpha}{\Gamma(\alpha+1)}\eta\right)^m\left[1-\left(\frac{T^\alpha}{\Gamma(\alpha+1)}\eta\right)\right]^{-1}\left\|x_1(t,x_0)-x_0\right\|$$

$$\leq \left(\frac{T^\alpha}{\Gamma(\alpha+1)}\eta\right)^m\left[1-\left(\frac{T^\alpha}{\Gamma(\alpha+1)}\eta\right)\right]^{-1}\frac{T^\alpha}{\Gamma(\alpha+1)}M\eta \quad (14)$$

But the inequality (7) is less than unity, then

$$\lim_{m\to\infty}\left(\frac{T^\alpha}{\Gamma(\alpha+1)}\eta\right)^m = 0 \quad (15)$$

Thus the right hand side of (14) equals to zero when $m \to \infty$. Suppose that $\varepsilon > 0$, we get a positive integer n such that $n < m$, and satisfied the next estimation for all m:

$\left\|x_{m+p}(t,x_0)-x_m(t,x_0)\right\| < \varepsilon$, p≥0.

Then according to the definition of uniformly convergent [1], we find that the sequence of function $\{x_m(t,x_0)\}_{m=0}^{\infty}$ is uniformly convergent to the function $x(t,x_0)$ and this function is continuous on the same interval.

Putting:

$$\lim_{m\to\infty} x_m(t,x_0) = x(t,x_0) \quad \textbf{(16)}$$

proof (iii):

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to prove $x(t,x_0)\in D_\alpha$, for all $t\in[0,T]$, $x_0\in D_{\alpha f}$ we assume that:

$$\left\|\frac{1}{\Gamma(\alpha)}\int_0^t\left[f\left(s,x_m(s,x_0),h(s)x_m(s,x_0),\int_{s-T}^{s}P(\tau,x_m(\tau,x_0),h(\tau)x_m(\tau,x_0))d\tau\right)\right](t-s)^{\alpha-1}ds-\right.$$

$$\left.-\frac{1}{\Gamma(\alpha)}\int_0^t\left[f\left(s,x(s,x_0),h(s)x(s,x_0),\int_{s-T}^{s}P(\tau,x(\tau,x_0),h(\tau)x(\tau,x_0))d\tau\right)\right](t-s)^{\alpha-1}ds\right\|$$

$$\leq\frac{\eta}{\Gamma(\alpha)}\int_0^t\|x_m(\tau,x_0)-x(\tau,x_0)\|(t-s)^{\alpha-1}ds$$

Then

$$\lim_{m\to\infty}\left\|\frac{1}{\Gamma(\alpha)}\int_0^t\left[f\left(s,x_m(s,x_0),h(s)x_m(s,x_0),\int_{s-T}^{s}P(\tau,x_m(\tau,x_0),h(\tau)x_m(\tau,x_0))d\tau\right)\right](t-s)^{\alpha-1}ds-\right.$$

$$\left.-\frac{1}{\Gamma(\alpha)}\int_0^t\left[f\left(s,x(s,x_0),h(s)x(s,x_0),\int_{s-T}^{s}P(\tau,x(\tau,x_0),h(\tau)x(\tau,x_0))d\tau\right)\right](t-s)^{\alpha-1}ds\right\|$$

$$\leq\lim_{m\to\infty}\frac{\eta}{\Gamma(\alpha)}\int_0^t\|x_m(\tau,x_0)-x(\tau,x_0)\|(t-s)^{\alpha-1}ds$$

Since the sequence $\{x_m(t,x_0)\}_{m=0}^{\infty}$ is uniformly convergent on $[0,T]$ to the function $x(t,x_0)$ on the same interval, then, we have

$$\lim_{m\to\infty}\left\|\frac{1}{\Gamma(\alpha)}\int_0^t\left[f\left(s,x_m(s,x_0),h(s)x_m(s,x_0),\int_{s-T}^{s}P(\tau,x_m(\tau,x_0),h(\tau)x_m(\tau,x_0))d\tau\right)\right](t-s)^{\alpha-1}ds=\right.$$

$$\left.=\frac{1}{\Gamma(\alpha)}\int_0^t\left[f\left(s,x(s,x_0),h(s)x(s,x_0),\int_{s-T}^{s}P(\tau,x(\tau,x_0),h(\tau)x(\tau,x_0))d\tau\right)\right](t-s)^{\alpha-1}ds\right\|$$

So $x(t,x_0)\in G_\alpha$, for all $x_0\in G_{\alpha f}$

3. Uniqueness of Solution

The study of the uniqueness solution of (1), will be introduced by the following:

Theorem 2.

Let all assumptions and conditions of theorem 1 be given then the problem (1), has a unique solution $x=x_\infty(t,x_0)$ on the domain (9).

Proof:

On the contrary, we suppose that there is another solution $\hat{x}(t,x_0)$ of the problem (1), which is defined by the following integral equation:

$$\hat{x}(t,x_0)=x_0+\frac{1}{\Gamma(\alpha)}\int_0^t\left[f\left(s,\hat{x}(s,x_0),h(s)\hat{x}(s,x_0),\int_{s-T}^{s}P(\tau,\hat{x}(\tau,x_0),h(\tau)\hat{x}(\tau,x_0))d\tau\right)\right](t-s)^{\alpha-1}ds \quad (17)$$

Now we shall prove that $\hat{x}(t,x_0)=x(t,x_0)$ for all $x_0\in D_{\alpha f}$, and to do this we need to prove the following inequality:

$$\|\hat{x}(t,x_0)-x_m(t,x_0)\|\leq\omega^{m+1}M^*\eta^m \quad (18)$$

where $\|f(t,x(t),h(t)x(t),g(t))\|\leq M^*$,

$\eta=(1+N)L$.

Let when m=0 in (8) and from (17) we find:

$$\|\hat{x}(t,x_0)-x_0\|=\left\|x_0+\frac{1}{\Gamma(\alpha)}\int_0^t\left[f\left(s,\hat{x}(s,x_0),h(s)\hat{x}(s,x_0),\int_{s-T}^{s}P(\tau,\hat{x}(\tau,x_0),h(\tau)\hat{x}(\tau,x_0))d\tau\right)\right](t-s)^{\alpha-1}ds-x_0\right\|$$

$$\leq\frac{1}{\Gamma(\alpha)}\int_0^t\left\|f\left(s,\hat{x}(s,x_0),h(s)\hat{x}(s,x_0),\int_{s-T}^{s}P(\tau,\hat{x}(\tau,x_0),h(\tau)\hat{x}(\tau,x_0))d\tau\right)\right\|(t-s)^{\alpha-1}ds$$

$$\leq\frac{1}{\Gamma(\alpha)}\int_0^t M^*(t-s)^{\alpha-1}ds$$

$$\leq\frac{t^\alpha}{\Gamma(\alpha+1)}M^* \quad , \quad t\in[0,T]$$

$$\|\hat{x}(t,x_0)-x_0\|\leq\frac{T^\alpha}{\Gamma(\alpha+1)}M^*$$

and when m=1 in (8) and from (17) we find that

$$\|\hat{x}(t,x_0)-x_1(t,x_0)\|=\left\|x_0+\frac{1}{\Gamma(\alpha)}\int_0^t\left[f\left(s,\hat{x}(s,x_0),h(s)\hat{x}(s,x_0),\int_{s-T}^{s}P(\tau,\hat{x}(\tau,x_0),h(\tau)\hat{x}(\tau,x_0))d\tau\right)\right](t-s)^{\alpha-1}ds-\right.$$

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$$-x_0-\frac{1}{\Gamma(\alpha)}\int_0^t\left[f\left(s,x_0(s,x_0),h(s)x_0(s,x_0),\int_{-\infty}^{s}P(\tau,x_0(\tau,x_0),h(\tau)x_0(\tau,x_0))d\tau\right)\right](t-s)^{\alpha-1}ds\Bigg\|$$

$$\leq\frac{1}{\Gamma(\alpha)}\int_0^t\left[(1+N)L\|\hat{x}(\tau,x_0)-x_0(\tau,x_0)\|-(1+N)L\|\hat{x}(\tau,x_0)-x_0(\tau,x_0)\|\right](t-s)^{\alpha-1}ds$$

$$=\frac{1}{\Gamma(\alpha)}(1+N)\int_0^t\left[L\|\hat{x}(\tau,x_0)-x_0(\tau,x_0)\|\right](t-s)^{\alpha-1}ds$$

$$\leq\eta\omega^2M^*$$

$$=\eta\omega^2M^*$$

therefore

$$\|\hat{x}(t,x_0)-x_1(t,x_0)\|\leq\eta\omega^2M^*$$

We find that the inequality (18) is satisfying when m=0, 1, 2.

Suppose that the inequality (18) is satisfying when m=p as the following inequality:

$$\|\hat{x}(t,x_0)-x_p(t,x_0)\|\leq\omega^{p+1}M^*\eta^p \qquad (19)$$

now

$$\|\hat{x}(t,x_0)-x_{p+1}(t,x_0)\|=\Bigg\|x_0+\frac{1}{\Gamma(\alpha)}\int_0^t\left[f\left(s,\hat{x}(s,x_0),h(s)\hat{x}(s,x_0),\int_{-\infty}^{s}G(s,\tau)P(\tau,\hat{x}(\tau,x_0),h(\tau)\hat{x}(\tau,x_0))d\tau\right)\right](t-s)^{\alpha-1}ds-$$

$$-x_0-\frac{1}{\Gamma(\alpha)}\int_0^t\left[f\left(s,x_p(s,x_0),h(s)x_p(s,x_0),\int_{s-T}^{s}P(\tau,x_p(\tau,x_0),h(\tau)x_p(\tau,x_0))d\tau\right)\right](t-s)^{\alpha-1}ds\Bigg\|$$

$$\leq\frac{1}{\Gamma(\alpha)}\int_0^t\left[(1+N)L\|\hat{x}(\tau,x_0)-x_p(\tau,x_0)\|-(1+N)L\|\hat{x}(\tau,x_0)-x_p(\tau,x_0)\|\right](t-s)^{\alpha-1}ds$$

$$=\frac{1}{\Gamma(\alpha)}(1+N)\int_0^t\left[L\|\hat{x}(\tau,x_0)-x_p(\tau,x_0)\|\right](t-s)^{\alpha-1}ds$$

$$\leq(1+N)L\eta^p\omega^{p+2}M^*$$

$$=\eta^{p+1}\omega^{p+2}M^*$$

then

$$\|\hat{x}(t,x_0)-x_{p+1}(t,x_0)\|\leq\omega^{p+2}M^*\eta^{p+1}$$

Thus we find that the inequality (18) is satisfying when m=0,1,2,… .

Then by a condition (16) we get:

$$\hat{x}(t,x_0)=\lim_{m\to\infty}x_m(t,x_0)=x(t,x_0)$$

and this proves that the two solutions are congruent in the domain (9).

4. Numerical Results

In this section we investigate the existence, uniqueness solution of fractional Integro- differential equations by using a new written program in Mat-Lab software. An example which is corresponding to our general form (3) , solved by using in Mat-Lab Release R2017a .

Example4.1: First, we consider the following fractional integro- differential equation, for $t \in I = [0,1]$.

$$\mathbf{D^{(\alpha)}u(t)=u(t)+\frac{8}{3\Gamma(0.5)}t^{1.5}-t^2-\frac{1}{3}t^3+\int_0^t u(s)ds \quad ,u(0)=0}$$

The exact solution and successive approximation method are show in figure(1)

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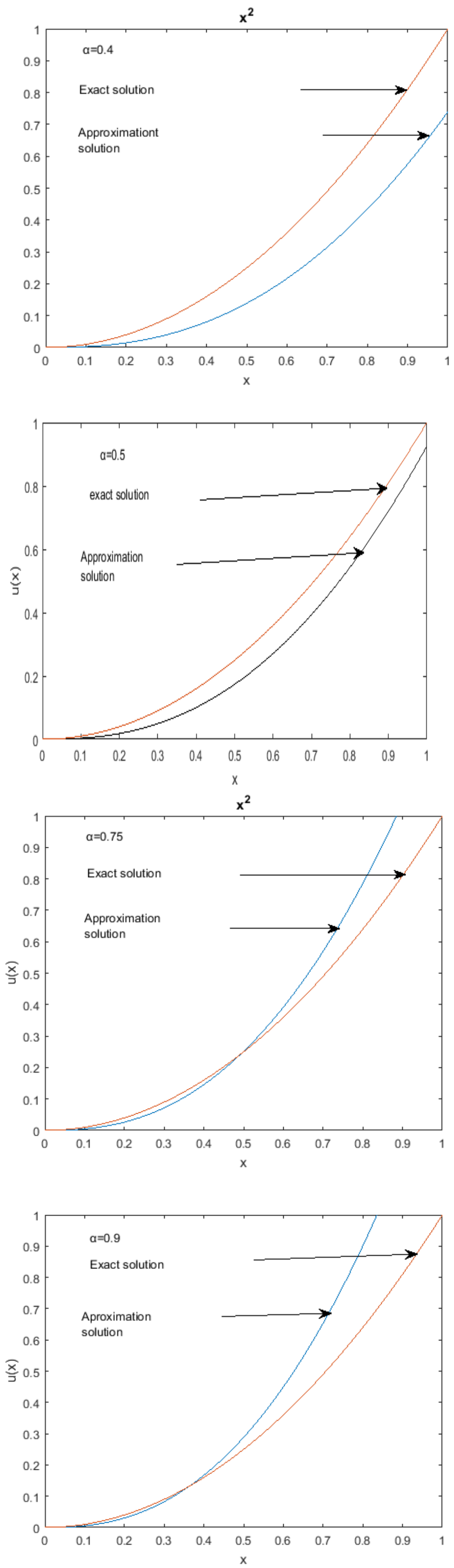
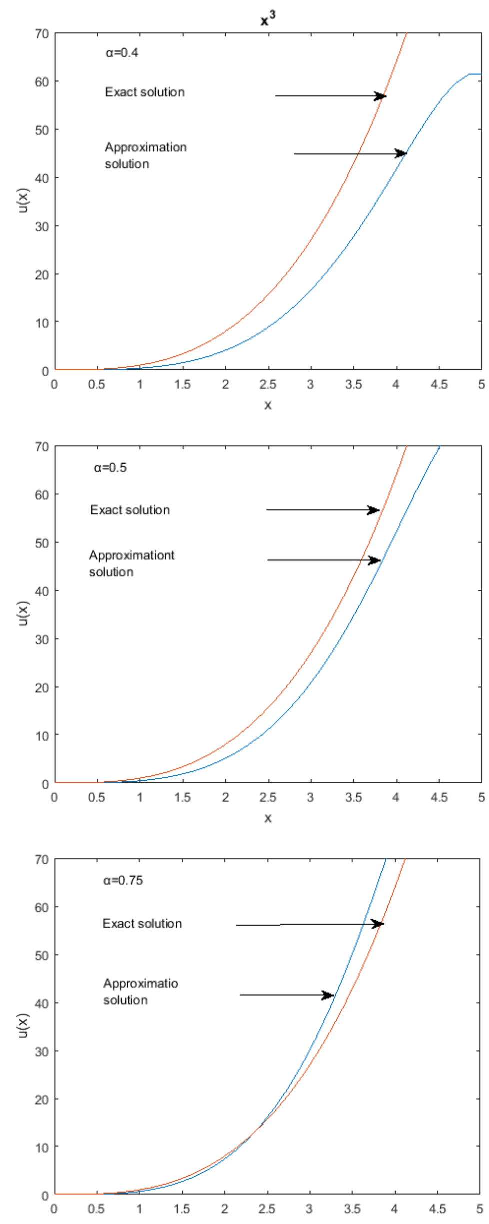


Figure(1) The exact solution is compared with the successive approximate in different value of (α)

Example4-2.: consider the following fractional integro-differential equation, for t ∈ I = [0,1].

$$\mathbf{D^{(\alpha)}u(t) = \frac{-t^2e^t}{5} + \frac{6t^{2.25}}{\Gamma(3.25)} + \int_0^t e^t u(s)ds \qquad , u(0) = 0}$$

The exact solution and successive approximation method are show in figure (2)



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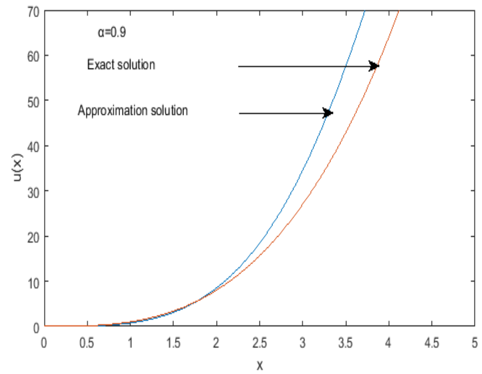
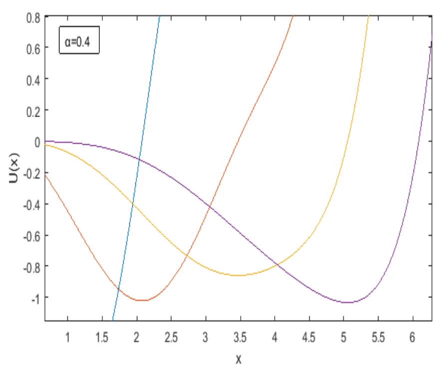


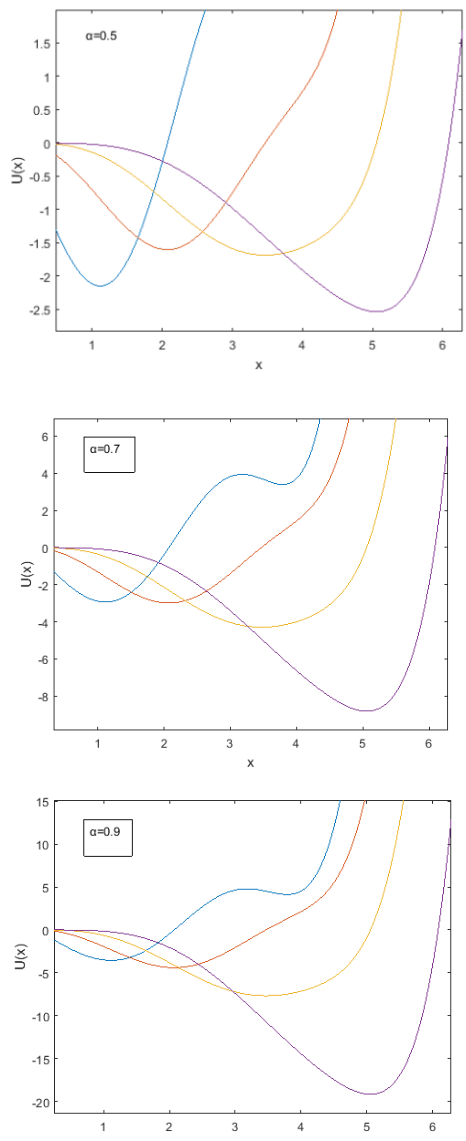
Figure(2) The exact solution is compared with the successive approximate in different value of (α)

Example 4-3.: consider the following fractional integro-differential equation, for t ∈ I = [0,1].

$$\mathbf{D^{(\alpha)}u(t) = (\cos t - \sin t) + \frac{2t^{1.5}}{\Gamma(2.5)} + \frac{t^{0.5}}{\Gamma(1.5)} + t(2 - 3\cos t - t\sin t + t^2 \cos t) + \int_{t-2\pi}^{t} s * \sin s\, ds \quad , u(0) = 0}$$

The successive approximation method with different value (α) of are show in the figure (3)





Figure(3) The successive approximate solution of Example4-3 in different value of (α)

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