Successive Approximation Method of Integro–Differential Equation With Applications

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ABSTRACT

The aim of this paper is studying the existence and uniqueness solution of integro- differential equations by using Successive approximations method of picard. The results of written program in Mat-Lab show that the method is very interested and efficient with comparison the exact solution for solving of integro-differential equation.

Keywords: Integro-differential equations, Successive approximations method, Mat-lab program method

1. Introduction

Integro-differential equations has been arisen in many mathematical and engineering fields, so that solving this sort of problems are more effective and beneficial in many research branches [4,6,7]. Analytical solution of this type of equation is not accessible in general form of equation and we can only obtain an exact solution only in distinct cases. But in industrial problems we have not spatial cases so that we try to solve this kind of equations numerically in general format. Many numerical sketches are employed to give an approximate solution with sufficient accuracy [8, 9, 10,11, 12] .

Definition1[7]. Let $\{f_m(t)\}_{m=0}^{\infty}$ be a sequence of functions which defined on a set $E \subseteq R^1$. We say that the sequence $\{f_m(t)\}_{m=0}^{\infty}$ converges uniformly to the limit function f on set E if, given $\varepsilon > 0$, there exists a positive integer N such that: $|f_m(t) - f(t)| < \varepsilon$, ($m \geq N, t \in E$) .

Definition2[7]. Assume that a continuous function f defined on $\Omega = \{(t,x): a \leq t \leq b, c \leq x \leq d\}$. The function f satisfied a Lipschitz condition in the variable x on Ω, provided that a constant $L > 0$ exists with the property that $|f(t,x_1) - f(t,x_2)| \leq L|x_1 - x_2|$,
for all $(t,x_1),(t,x_2) \in \Omega$. And the constant L is called a Lipschitz constant for f.

Definition3[7]. A solution $x(t)$ is stable if for each $\varepsilon > 0$, there exists a $\delta > 0$ such that any solution $\bar{x}(t)$ which satisfies $\|\bar{x}(t_0) - x(t_0)\| < \delta$ for some t_0, also satisfies $\|\bar{x}(t) - x(t)\| < \varepsilon$ for all $t \geq t_0$.

Butris[3] investigate the existence and uniqueness solution by using the successive approximation method for the following problem:-

$$x(t) = x_0 + f(t) + \int_{-\infty}^{t} G(t,\tau)g\big(\tau, x(\tau)\big)d\tau\,,$$

where, $x \in D, D$ is a bounded and closed domain subset of Euclidean R^n.

In this paper, we study the following problem:-

$$\frac{dx}{dt} = \lambda f\big(g(t,x)\beta(t,\tau)\big), \quad \dots \; (P)$$

where, $\beta(t,\tau) = \int_{-\infty}^{t} K(t,\tau)d\tau\,, \lambda > 0,$ D is closed and bounded domain subset of Euclidean R^n.

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Suppose that the functions $g(t,x)$ and $f(g(t,x)\beta(t,\tau))$ are defined and continuous in the domain

$$\Omega = \{0 \le t \le b, x \in D\}. \quad \dots \ (1.1)$$

Furthermore, $K(t,\tau)$ is $singulaer\ kernal$, provided that

$$\|K(t,\tau)\| \le \delta e^{-\gamma(t-\tau)}, \ \ \delta, \gamma > 0.$$

Suppose that the vector functions $g(t,x)$ and $f(g(t,x)\beta(t,\tau))$ are satisfying the following inequalities:

$$\|f(g(t,x)\beta(t,\tau))\| \le M, \quad \dots \ (1.2)$$

$$\|g(t,x_1) - g(t,x_2)\| \le L\|x_1 - x_2\|, \quad \dots \ (1.3)$$

$$\| f((g(t,x_2)\beta(t,\tau))\| \le \beta(t,\tau) \ \|g(t,x_1) - g(t,x_2)\|$$

$$\le L\frac{\delta}{\gamma}\|x_1 - x_2\|, \quad \dots \ (1.4)$$

for each $t \in [0,b]$, $x, x_1, x_2 \in D$, where M and L are positive constants and $\|.\| = max|.|$.

Now we define the non-empty set by:

$\Omega_f = \Omega - \lambda bM$.

The condition q is less than unity, i. e.

$$q = \lambda L\frac{\delta}{\gamma}b < 1. \quad \dots \ (1.5)$$

Suppose, $\{x_m(t)\}_{m=0}^{\infty}$ be a sequence of functions which define on the interval $[0,b]$ by:

$$x_m(t) = x_0 + \lambda\int_0^t f(g(\tau, x_{m-1}(\tau)) \ \beta(t,\tau))]d\tau \quad \dots \ (1.6)$$

with

$x_0(0) = x_0, \quad m = 1,2,3, \dots$.

2. Existences Solution of (P).

The investigation of existence solution of the problem (P) will be introduced by the following theorem.

Theorem 1. (Existence theorem). Let the functions $g(t,x)$ and $f(g(t,x)\beta(t,\tau))$ are defined, continuous on the domain (1.1). Suppose that the functions $g(t,x)$ and $f(g(t,x)\beta(t,\tau))$ are satisfying the inequalities (1.3) and conditions (1.4), (1.5). Thus there exist a sequence of functions (1.6) for which is convergent uniformly on the domain

$$(t, x_0) \in [a,b] \times \Omega_f, \quad \dots \ (2.1)$$

to the limit function $x(t)$ which is a continuous on the domain (2.1) and satisfies the following integral equations:-

$$x(t) = x_0 + \lambda\int_0^t f\left(g(\tau, x(\tau))\beta(t,\tau)\right) d\tau$$

and it's a unique solution of the problem (P), provided that:-

$$\|x(t) - x_0\| \le M\lambda b, \quad \dots \ (2.3)$$

$$\|x_{m+1}(t) - x_m(t)\| \le q^{m+1}(1-q)^{-1}M, \quad \dots \ (2.4)$$

for all $t \in [0,b]$ and $x_0 \in \Omega_f$, m=0,1,2,… .

When $m = 1$ in (1.6), we get

$$\|x_1(t) - x_0\| \le \left\|\lambda\int_0^t f\left[.g(t,x_0)\int_{-\infty}^t K(t,\tau)\,d\tau\right]d\tau\right\| \le \lambda Mb$$

So $x_1(t) \in \Omega \ for\ all\ x_0 \in \Omega_f$.

When $m = 2$ in (1.6), we get

$$\|x_2(t) - x_0\| \le \left\|\lambda\int_0^t f\left[g(t,x_0)\int_{-\infty}^t K(t,\tau)\,d\tau\right]d\tau\right\| \le \lambda Mb,$$

$$\|x_2(t) - x_0\| \le \lambda MT,$$

Then, $x_2\left(g(t,x(t))\right) \in \Omega, for\ all\ x_0 \in \Omega_f, t \in [0,b]$.

By mathematical indication, obtain that

$$\|x_m(t) - x_0\| \le \left\|\lambda\int_0^t f\left[g(t,x_{m-1}(\tau))\int_{-\infty}^t \Omega(t,\tau)x_{m-1}(\tau)d\tau\right]d\tau\right\| \le \lambda Mb.$$

That is

$$\|x_m(t) - x_0\| \le \lambda Mb \quad \dots \ (2.5).$$

Then, $x_m(t) \in \Omega, for\ all\ x_0 \in \Omega_f, t \in [0,b]$.

Next, we shall prove that the sequence of functions $\{x_m(t)\}_{m=0}^{\infty}$ is convergent uniformly to the limit function $x(t)$ on the interval $[0,b]$, for each $m = 0,1,2, \dots$.

By mathematical indication, we can prove that

$$\|x_{m+1}(t) - x_m(t)\| \le Mq^{m+1}(1-q)^{-1}M \quad \dots \ (2.6)$$

for all m $= 0,1,2, \dots$, and $x_0 \in \Omega_f, 0 \le t \le b$.

Thus by taking the summation two sides of the inequality (2.6) can be written as:

$$\sum_{m=0}^{\infty}\|x_{m+1}(t) - x_m(t)\| \le M\sum_{m=0}^{\infty} q^{m+1}(1-q)^{-1}M,$$

Since $q < 1$, so the sequence $\sum_{m=0}^{\infty} q^{m+1}(1-q)^{-1}M$ is a geometric series which is convergent on $[0,b]$.

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Then $\sum_{m=0}^{\infty}\|x_{m+1}(t)-x_m(t)\|$ is absolutely and uniformly convergent on $[0,b]$.

But

$x_m(t)=x_0+\sum_{k=0}^{m-1}\|x_{k+1}(t)-x_k\,(t)\|$,

for which implies that the sequence of functions $\{x_m(t)\}_{m=0}^{\infty}$ on $[0,b]$ is convergent uniformly to the limit function $x(t)$, i. e.

$$\lim_{m\to\infty} x_m(t)=x(t). \quad \ldots \ (2.7)$$

Since the sequence of functions $\{x_m(t)\}_{m=0}^{\infty}$ on $[0,b]$ is continuous. Then the limiting function is also continuous on $[0,b]$.

We shall prove the equality:-

$$\lim_{m\to\infty}\int_0^t f(g(\tau,x_m(\tau)))\beta(t,\tau))\ d\tau$$

$$=\int_0^t f(g(\tau,x(\tau)))\beta(t,\tau)\ )d\tau$$

Taking

$\left\|\lambda\int_0^t f(g(\tau,x_m(\tau))\beta(t,\tau))d\tau-\right.$

$\left.\lambda\int_0^t f(g(\tau,x(\tau))\beta(t,\tau))d\tau\right\|\le\lambda L\frac{\delta}{\gamma}b\|x_m(t)-x(t)\|$

But, $\{x_m(t)\}_{m=0}^{\infty}$ converges uniformly, so

$\left\|\lambda\int_0^t f(g(\tau,x_m(\tau))\beta(t,\tau))d\tau-\right.$

$\left.\lambda\int_0^t f(g(\tau,x(\tau))\beta(t,\tau))d\tau\right\|\le\lambda L\frac{\delta}{\gamma}b\in_1,\ \in_1>0,$

Choosing, $\in_1=\frac{\gamma\in}{\delta Lb\lambda}$, hence

$\left\|\lambda\int_0^t f(g(\tau,x_m(\tau))\beta(t,\tau))d\tau-\right.$

$\left.\lambda\int_0^t f(g(\tau,x(\tau))\beta(t,\tau))d\tau\right\|\le\epsilon.$

Then

$$\lim_{m\to\infty}\int_0^t f(g(\tau,x_m(\tau)))\beta(t,\tau))\ d\tau$$

$$=\int_0^t f(g(\tau,x(\tau)))\beta(t,\tau)\ )d\tau.$$

So $x(t)$ is a solution of integro- differential equation (P).

3. Uniqueness solution of $(\boldsymbol{P})$.

This section contains the uniqueness solution of Integro-differential equations of $(P\ )$.

Theorem2.(Uniqueness theorem). Suppose that problem (P) was defined in domain (1.1) and satisfy inequalities, conditions in previous section. Then the solution is a unique .

Proof. From the following

$x(t)=x_0+$

$\lambda\int_0^t f\big(g(t,x(\tau))\beta(t,\tau)\ \big)d\tau.$

We assume that $\mathrm{y}(t)$ is separate solution from, (P), that is

$\mathrm{y}(t)=x_0+$

$\lambda\int_0^t f\big(g(t,\mathrm{y}(\tau))\beta(t,\tau)\ \big)d\tau$

Then

$\|x(t)-\mathrm{y}(t)\|\ \le\lambda\int_0^t\Big\|f\Big(g\big(t,x(\tau)\big)\beta(t,\tau)\Big)-$

$f\Big(g\big(t,\mathrm{y}(\tau)\big)\beta(t,\tau)\Big)\Big\|\ d\tau$

$\le\lambda\int_0^t\frac{\delta}{\gamma}\big\|g\big(t,x(\tau)\big)-\big(g\big(t,\mathrm{y}(\tau)\big)\big\|\|\beta(t,\tau)\|\ d\tau$

$\le\lambda\int_0^t\frac{\delta}{\gamma}L\|x(\tau)-\mathrm{y}(\tau)\|d\tau,$

and hence

$\|x(t)-\mathrm{y}(t)\|\le q\|x(t)-\mathrm{y}(t)\|$

By the condition (1.5), we have $\|x(t)-\mathrm{y}(t)\|<\|x(t)-\mathrm{y}(t)\|$, that is contradiction to the assumption above. Thus $\|x(t)-\mathrm{y}(t)\|=0$. So $x(t)=\mathrm{y}(t)$. Thus $\mathrm{x}(t)$ is a unique solution of (P).

4. Numerical results in successive approximation method.

In this section, investigating the existence, uniqueness and stability solution of Integro-differential equations by using a new written program in Mat-Lab software is done. An example which is corresponding to our general form, (P) solved by method of Picard iterations in Mat-Lab.

Example1 . Consider the following problem:-

$$\frac{dx}{dt}=a\sin(t-x)\int_{-\infty}^{t}be^{-c(t-\tau)}d\tau$$

$x(0)=0.4,$ $where\ \alpha,\delta$ and γ are positive constants.

For the initial condition $x(0)=0.4$,we made a Mat-Lab program to solve the integro-differential equations that contains the plot of family of solutions of problem (P). From the results, investigate that our approximation

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result is corresponding to the exact algebraic result.

Defined a sequence of functions $\{x_m(t)\}_{m=0}^{\infty}$ by:

$x_m(t) = x_0 + \int_0^t [\text{a} \sin(t - x_{m-1}(\tau)) \int_{-\infty}^{t} be^{-c(t-\tau)} d\tau]\, d\tau,$

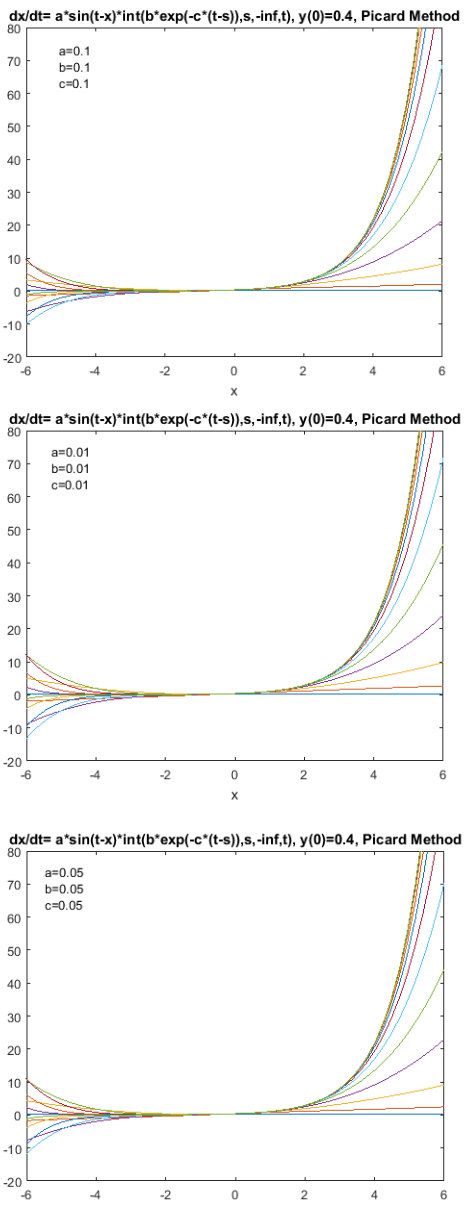
$x_0(0) = x_0, m = 1,2, \dots .$

When $m = 0$, we have

$x_1(t) = 0.4 + \int_0^t [\text{a} \sin(\tau - 0.4) \int_{-\infty}^{t} be^{-c(\,t-\tau)} d\tau]\, d\tau,$

where, $x_0(0) = 0.4$.

With programming simplicity we will get the following grap



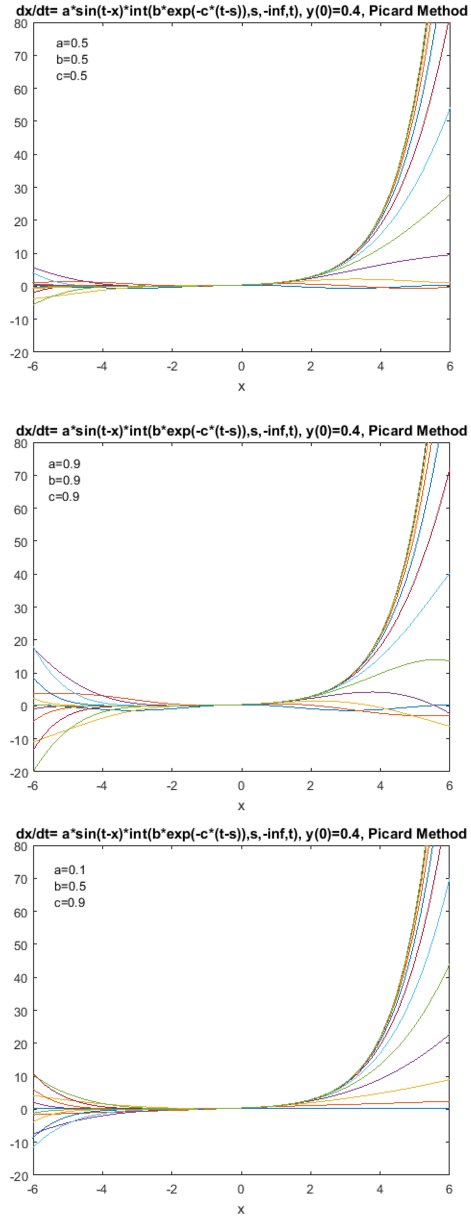


Figure1:- Picard Solution curves for different values of a, b, c in example (1).

Example2. Apply the Picard iteration for the following problem:

$$\frac{dx}{dt} = \alpha\left(x^2(t) + x(t)t - \beta\int_0^t (x^2(\tau) + x(\tau)\,\tau\,)d\tau - 0.16\right),$$

$x(0) = 0.4,$ *where* $\alpha$and β are positive constants.

Define a sequence of functions $\{x_m(t)\}_{m=0}^{\infty}$ by:-

$$x_m(t) = x_0 + \alpha\int_0^t [\; x_m^2\;(\tau) + \tau\, x_m\;(\tau)$$

$$-\beta\int_0 \left(x_m^2\;(\tau) + \tau\;x_m\;(\tau)\right) d\tau - 0.16]\, d\tau$$

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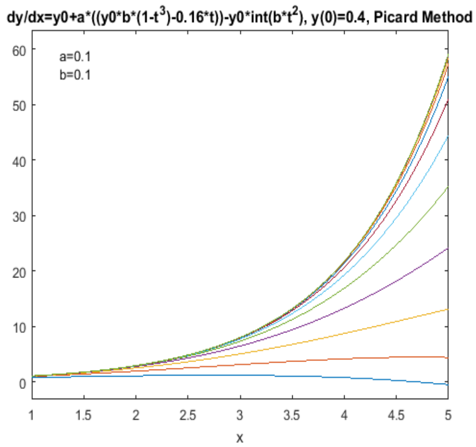
$x_0(0) = \ x_0, m = 1,2,3, \dots .$

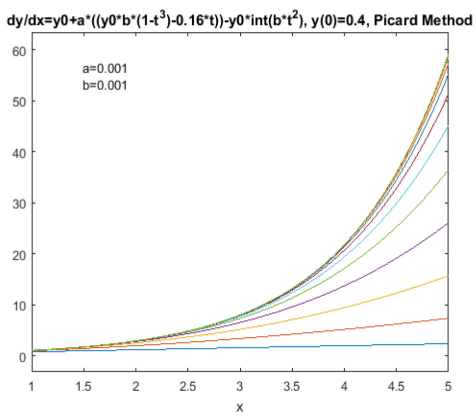
When $m = 1$, we find

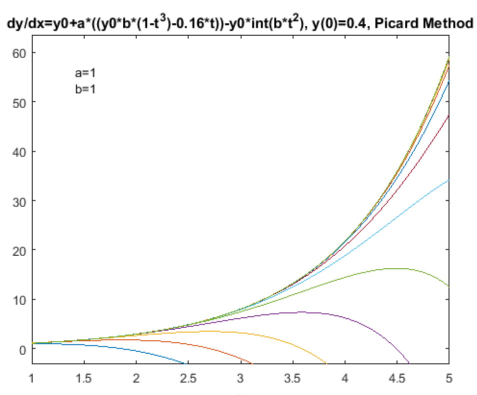
$$x_1(t) = x_0 + \alpha \int_0^t \left[\ x_0^2 \ + \tau\, x_0 \ - \beta \int_0^t (x_0^2 \ + \tau\, x_0 - 0.16\ )\ d\tau \right] d\tau,$$

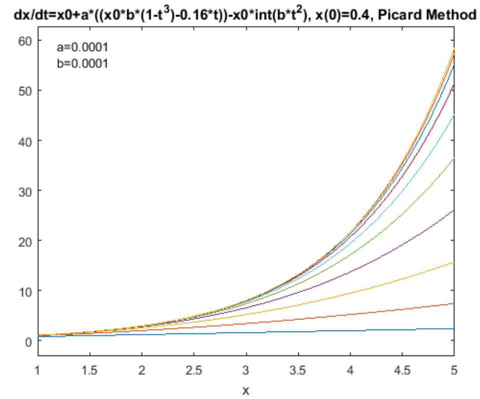
with

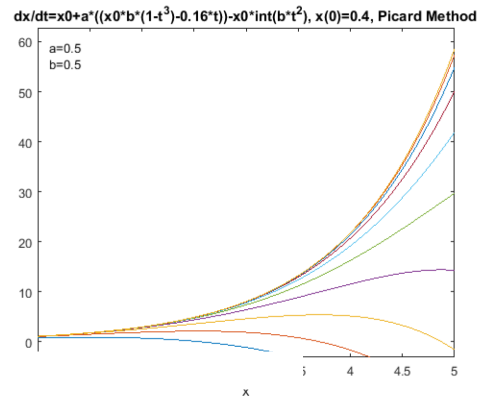
$$x_0(0) = 0.4$$

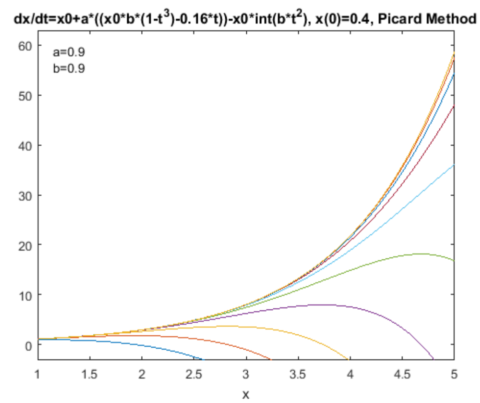


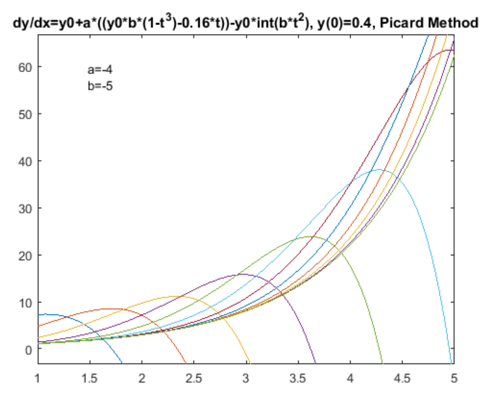












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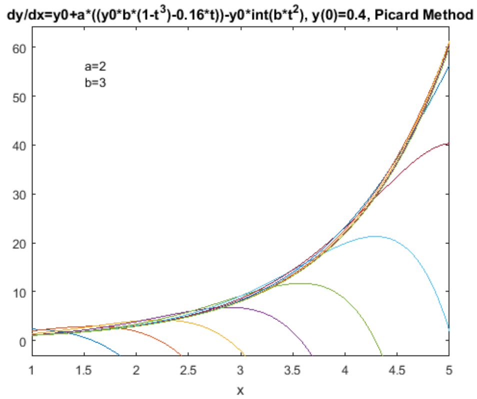

Figure(2): Picard Solution curves for different values of a, b where $a = \alpha$, $b = \beta$ in example (2).

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