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Some Properties of Γ-supercyclic operators

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ABSTRACT

In this paper, we define Γ-transitive operators to study some further properties of Γ-supercyclic operators. In particular, we study Γ-supercyclic criterion which is a set of some sufficient conditions for an operator to be Γ-supercyclic. We use these conditions to partially answer an open problem in the literature. In particular, we show that if $\Gamma = r\mathbb{D}$ and T satisfies Γ-criterion, then T is supercyclic.

Keywords: Supercyclic operator, diskcyclic operators, Γ-supercyclic operators.

1. Introduction

An operator T is called hypercyclic if there is a vector $x \in X$ such that $Orb(T, x) = \{T^n x : n \in \mathbb{N}\}$ is dense in X, such a vector x is called hypercyclic for T . The notion of hypercyclic operator comes from the much older notion of cyclic operators. The first example of hypercyclic operators in a Banach space was constructed by Rolewicz (Rolewicz, 1969). He proved that if B is a backward shift on the Banach space $l^p(\mathbb{N})$ then λB is hypercyclic for any complex number $\lambda, |\lambda| > 1$. This led the authors in (Hilden and Wallen, 1974) to consider the scaled orbit of an operator. An operator T is supercyclic if there is a vector $x \in X$ such that $\mathbb{C}Orb(T, x) = \{\lambda T^n x : \lambda \in \mathbb{C},\ n \in \mathbb{N}\}$ is dense in X, where x is called supercyclic vector. Also, an operator T is called diskcyclic if there is a vector $x \in X$ such that the disk orbit $\mathbb{D}Orb(T, x) = \{\lambda T^n x : \lambda \in \mathbb{C}, |\lambda| \le 1,\ n \in \mathbb{N}\}$ is dense in X such a vector x is called diskcyclic for T (Zeana, 2002). For more information on these concepts, one may refer to (Bayart and Matheron, 2009; Bamerni and Kilicman, 2016)
We recall the following Lemma from the literature, which is needed in the main section.

Lemma 1.1. (Bamerni and Kilicman, 2015)
Let $T \in B(X)$. Suppose that $\{n_k\}_{k\in\mathbb{N}}$ is an increasing sequence of positive integers,
$D_1, D_2 \in X$ are two dense sets and $S: D_1 \to D_2$ a map such that

1. $\|T^{n_k}x\|\|S^{n_k}y\| \to 0$ for each $x \in D_1$ and $y \in D_2$,
2. $\|S^{n_k}y\| \to 0$ and $T^{n_k}S^{n_k}y \to y$ for each $y \in D_2$.

Then T is said to be satisfied diskcyclic Criterion, and T is diskcyclic operator.
For a non-empty subset Γ of the complex plane $\mathbb{C}$, an operator T is called Γ-supercyclic if there exists $x \in X$ such that $\Gamma Orb(T, x) = \{\lambda T^n x : \lambda \in \Gamma,\ n \in \mathbb{N}\}$ is dense in X (Charpentier et al, 2016). It is clear from Γ-supercyclicity notion that:

1. If $\Gamma = \{1\}$, then Γ-supercyclic is hypercyclic,
2. If $\Gamma = \mathbb{C}$, then Γ-supercyclic is supercyclic,
3. If $\Gamma = \mathbb{D}$, then Γ-supercyclic is diskcyclic,

In 2019, the author in (Abbar, 2019) studied Γ-supercyclicity for strongly continuous semigroups. In particular, he characterized a set Γ in which Γ-supercyclicity for strongly continuous semigroups coincides to hypercyclicity.
In the main section, we give some characterizations to Γ-supercyclic operators. In particular, we define Γ-transitive operators. Then, we show that an operator T is Γ-supercyclic if and only if T is Γ-transitive. Also, we give some sufficient conditions for an operator to be Γ-supercyclic which is called Γ-supercyclic criterion. The authors in (Charpentier et al, 2016) ask for which sets Γ, Γ-supercyclicity equivalent to supercyclicity, he gives a partial answer to this question when $\sigma_p(T^*)$ is non-empty. Therefore, we give another partial answer to this problem. In particular, we show that if $\Gamma = r\mathbb{D}$ and T satisfies Γ-criterion, then T is supercyclic.
it changed during time and what were the factors affecting it.

2. MAIN RESULTS

In this paper, all Banach spaces are infinite dimensional separable over the field $\mathbb{C}$ of complex numbers. The set of all bounded linear operators on a Banach space is denoted by $B(X)$. Also, we denote set of all Γ-supercyclic operators on a Banach space by ΓSC(X) and the set of all Γ-supercyclic vectors for an operator T by ΓSC(T).

Definition 2.1.
A bounded linear operator $T \in B(X)$ is called Γ-transitive if for any pair U, V of nonempty open subsets of X, there exist $\alpha \in \Gamma$, and $n \geq 0$ such that $T^n(\alpha U) \cap V \neq \emptyset$ or equivalently, there exist $\alpha \in \Gamma^c$, and $n \geq 0$ such that $T^{-n}(\alpha U) \cap V \neq \emptyset$.

Proposition 2.2.
Let $T_1, T_2 \in B(X)$ such that $T_1 T_2 = T_2 T_1$ and the range of T_2 $(R(T_2))$ be a dense set in X. If $x \in \Gamma SC(T_1)$, then $T_2 x \in \Gamma SC(T_1)$.
Proof
Since $x \in \Gamma SC(T_1)$, then $\overline{\Gamma Orb(T_1, x)} = \overline{\{\lambda T_1^n x: \lambda \in \Gamma,\ n \geq 0\}} = X$ and

$$\begin{aligned}\overline{\Gamma Orb(T_1, T_2 x)} &= \overline{\{\lambda T_1^n T_2 x: \lambda \in \Gamma, n \geq 0\}}\\ &= \overline{\{\lambda T_2 T_1^n x: \lambda \in \Gamma, n \geq 0\}}\\ &\supseteq T_2\left(\overline{\{\lambda T_1^n x: \lambda \in \Gamma, n \geq 0\}}\right)\\ &= T_2(X) = R(T_2).\end{aligned}$$

Thus, $\Gamma Orb(T_1, T_2 x)$ is dense in X and hence $T_2 x \in \Gamma SC(T_1)$.

Corollary 2.3
If x is Γ-supercyclic vector for T then for all $n \in \mathbb{N}$, $T^n x$ is Γ-supercyclic vector for T.

Corollary 2.4.
The set ΓSC(T) is dense in X.

Proposition 2.5.
Every Γ-supercyclic operator on X is Γ-transitive.
Proof
Let T be a Γ-supercyclic operator, then, ΓSC(T) is dense. Let U and V be two open sets, then there exist an $\alpha \in \Gamma$ and a non-negative integer p such that $\alpha T^p x \in U$. Now, we can choose $\lambda \in \Gamma$ and $n \geq p$ such that $\lambda/\alpha \in \Gamma$ and $\lambda T^n x \in V$. Thus $T^{n-p}(\lambda/\alpha)U \cap V \neq \emptyset$ and so T is Γ-transitive.

Proposition 2.6.
Every Γ-transitive operator is Γ-supercyclic and

$$\Gamma SC(T) = \bigcap_k \left(\bigcup_{\substack{\lambda \in \Gamma^c \\ n \in \mathbb{N}}} T^{-n}(\lambda B_k) \right)$$

is a dense G_δ set, where $\{B_k\}$ is a countable open basis for X.
Proof
We have $x \in \Gamma SC(T)$ if and only if the set $\{\lambda T^n x: \lambda \in \Gamma,\ n \geq 0\}$ is dense in X if and only if for each $k > 0$, there exist $\lambda \in \Gamma$ and $n \in \mathbb{N}$ such that $\lambda T^n x \in B_k$ if and only if

$$x \in \bigcap_k \left(\bigcup_{\substack{\lambda \in \Gamma^c \\ n \in \mathbb{N}}} T^{-n}(\lambda B_k) \right)$$

If and only if

$$\Gamma\mathrm{SC(T)} = \bigcap_k \left(\bigcup_{\substack{\lambda\in\Gamma^c \\ n\in\mathbb{N}}} T^{-n}(\lambda B_k) \right).$$

Since ΓSC(T) can be written as a countable intersection of open sets, then ΓSC(T) is a G_δ set. Moreover, it follows from the Baire category theorem that ΓSC(T) is dense if and only if each open set $F_k = \bigcup_{\substack{\lambda\in\Gamma^c \\ n\in\mathbb{N}}} T^{-n}(\lambda B_k)$ is dense; i.e, if and only if for each non-empty open set U and any $k \in \mathbb{N}$ one can find $n \in \mathbb{N}$ and $\lambda \in \Gamma^c$ such that

$$U \cap\ T^{-n}(\lambda B_k) \neq \emptyset$$

Since $\{B_k\}$ is a countable open basis for X this is equivalent to the Γ-transitivity of T.
The following theorem gives some equivalent assertions to Γ-supercyclic.

Theorem 2.7.
Let $T \in B(X)$. The following statements are equivalent.

1. T ∈ ΓSC(X),
2. For each $x, y\ \in X$, there exist sequences $\{x_k\} \in X, \{n_k\} \in \mathbb{N}$ and $\{\lambda_k\} \in \Gamma$ such that $x_k \to x$ and $T^{n_k}\lambda_k x_k \to y$.
3. For each $x, y\ \in X$ and each neighborhood M of the zero in X, there exist $z \in X$, $n \in \mathbb{N}$ and $\lambda \in \Gamma$ such that $x - z \in M$ and $T^n\lambda z - y \in M$.

Proof

1 ⇒ 2: Let $x, y\ \in X$ and let $B_k = \mathbb{B}\left(x, \frac{1}{k}\right), C_k = \mathbb{B}\left(y, \frac{1}{k}\right)$ for all $k \geq 1$. From (1) and Proposition there exist sequences $\{x_k\} \in X, \{n_k\} \in \mathbb{N}$ and $\{\lambda_k\} \in \Gamma$ such that $x_k \in B_k$ and $T^{n_k}\lambda_k x_k \in C_k$ for all $k \geq 1$. Then, $\|x_k - x\| < 1/k$ and $\|T^{n_k}\lambda_k x_k - y\| < 1/k$ for all $k \geq 1$.

2 ⇒ 3: From the proof of the last part, if we take $z = x_k$for a large enough $k \in \mathbb{N}$.

3 ⇒ 1: Let U and V be two non-empty subsets of X. Let M be a neighborhood of zero, let $x \in U$ and $y \in V$. Then there exist $z \in X$, $n \in \mathbb{N}$ and $\lambda \in \Gamma$ such that $x - z \in M$ and $T^n\lambda z - y \in M$. It follows that $z \in U$ and $T^n\lambda z \in V$ and so $T^n\lambda U \cap V \neq \emptyset$.

Theorem 2.8.
Let $T \in B \in (X)$, and let $\{n_k\}_{k\in\mathbb{N}}$ be an increasing sequence of positive integers and $\{\lambda_{n_k}\}_{k\in\mathbb{N}} \in \Gamma$ such that there exists

1. A dense subset $D_1 \in X$ such that $\|\lambda_{n_k} T^{n_k} x\| \to 0$ for all $x \in D_1$.
2. A dense subset $D_2 \in X$ and a mapping $S: D_1 \to D_2$ such that $\|\lambda_{n_k}^{-1} S^{n_k} y\| \to 0$ and $T^{n_k}S^{n_k}y \to y$ for all $y \in D_2$.

Then T is said to be satisfied Γ-supercyclic Criterion with respect to the sequence $\{\lambda_{n_k}\}_{k\in\mathbb{N}}$and T is an Γ-supercyclic operator.

Proof

Let U and V be two open sets in X. Then we can find $x \in D_1 \cap U$ and $y \in D_2 \cap V$. By hypothesis, there exists a large positive integer k and a small positive integer ε such that
$\|\lambda_{n_k} T^{n_k} x\| < \varepsilon/2,\ \|\lambda_{n_k}^{-1} S^{n_k} y\| < \varepsilon/2$ and $\|T^{n_k}S^{n_k}y - y\| < \varepsilon/2$

Let $z = x + \lambda_{n_k}^{-1} S^{n_k} y$, then

$$\|z - x\| = \|\lambda_{n_k}^{-1} S^{n_k} y\| < \varepsilon/2$$

it follows that $z \in U$. Now, we have

$$\lambda_{n_k} T^{n_k} z = \lambda_{n_k} T^{n_k} x + T^{n_k} S^{n_k} y$$

Then

$$\begin{aligned} \|\lambda_{n_k} T^{n_k} z - y\| &= \|\lambda_{n_k} T^{n_k} x + T^{n_k} S^{n_k} y - y\| \\ &\leq \|\lambda_{n_k} T^{n_k} x\| + \|T^{n_k} S^{n_k} y - y\| \\ &\leq \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

It means that $\lambda_{n_k} T^{n_k} z \in V$, and so $\lambda_{n_k} T^{n_k} U \cap V \neq \emptyset$.

By **Definition 2.1.** and **Proposition 2.6.** T is Γ-supercyclic.

The authors in (Charpentier et al, 2016) ask for which sets Γ, Γ- supercyclicity equivalent to supercyclicity. He gives a partial answer to this question when $\sigma_p(T^*)$ is non-empty. Now, if $\Gamma = r\mathbb{D}$ for some $r \in \mathbb{C}; r \neq 0$ and T satisfies r𝔻-criterion, then the following theorem gives another partial answer to question 3 of (Charpentier et al, 2016).
First, we need the following definition.

Definition 2.9.
An operator T is called r𝔻-supercyclic for some positive integer r if there exists a vector x such that the set $\lambda T^n x: n \geq 0$, $0 \leq |\lambda| \leq r\}$ is dense in X. In this case the vector x is called r𝔻-supercyclic vector for T.

Theorem 2.10.
r𝔻-supercyclic criterion is equivalent to diskcyclic-criterion.
Proof
Suppose that T satisfies the hypothesis of **Theorem 2.8.** for $\Gamma = r\mathbb{D}$, then it is easy to show that $\|T^{n_k}x\|\|S^{n_k}y\| \to 0$ and $\|S^{n_k}y\| \to 0$ for all $x \in D_1$ and $y \in D_2$. It follows that the hypothesis of **Lemma 1.1.** satisfy.
Conversely, suppose that T satisfies the hypothesis of **Lemma 1.1.** Let $\{\varepsilon_k\}_{k\in\mathbb{N}}$ be a sequence of positive numbers converges to zero. Let $\{x_n\}_{n\in\mathbb{N}} \subset D_1$ and $\{y_n\}_{n\in\mathbb{N}} \subset D_2$ be two countable dense subsets in X. By hypothesis of **Lemma 1.1.**, for each $1 \leq i,j \leq k$ we have $\|S^{n_k}y_j\| < \varepsilon_k$, $T^{n_k}S^{n_k}y_j \to y_j$ and

$$\|T^{n_k}x_i\|\|S^{n_k}y_j\| < \varepsilon_k^2 \qquad (1)$$

Suppose that for each $k \geq 1$,

$$\lambda_{n_k} = \frac{r}{\varepsilon_k} \max_{1\leq j\leq k}\{\|S^{n_k}y_j\|\}$$

It follows that $\lambda_{n_k} \in r\mathbb{D}\backslash\{0\}$ for all k, and
$\frac{1}{\lambda_{n_k}}\|S^{n_k}y_j\| \leq \frac{\varepsilon_k}{r}$ for all $j \leq k$
By Equation (1), for all $i \leq k$, we have

$$\lambda_{n_k}\|T^{n_k}x_i\| = \frac{r}{\varepsilon_k} \max_{1\leq j\leq k}\{\|S^{n_k}y_j\|\}\|T^{n_k}x_i\| < r\varepsilon_k$$

Now, the proof follows when $k \to \infty$.

Corollary 2.11.
If T satisfies Γ-supercyclic criterion for $\Gamma = r\mathbb{D}$, then T is supercyclic operator.

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